Quadratic Voting with Vote Credits

Yassin Alaya *

June 2024

Abstract

A society faces K binary decisions to be decided through K simultaneous polls using Quadratic Voting (QV) with vote credits: Each voter is endowed with a budget of vote credits, which she uses to buy a number v_k of votes (positive or negative) on each poll k at a cost of v_k^2 vote credits per poll. I characterise voting equilibria as the number of voters tend to infinity and compare total welfare under QV with vote credits and majority voting under different preference distributions. Welfare under QV relies on overall preference intensity being uncorrelated with preferences on individual polls. Independent preferences and a large number of polls create equality in preference intensities and favours QV with credits. I present a pathological example where QV decides in favour of arbitrarily small preferences.

1 Introduction

A society faces a binary collective decision between alternatives A and B. When a majority of close to indifferent voters outnumbers a minority with opposite but more intense preferences, the welfare maximising decision may be to follow the minority's will. Under the standard system of majority voting, however, the majority's will is always implemented. The idea of Quadratic Voting (QV) as introduced by Posner and Weyl (2015) intends to solve this tyranny of the majority problem: Each voter chooses how many votes to buy (positive or negative) and pays a cost quadratic in the number of votes. If the vote total is positive, alternative A is chosen, if the vote total is negative, alternative B is chosen. The idea of a quadratic cost function is to induce voters to buy a number of votes approximately proportional to how much they value their preferred alternative being chosen. Thus, in a quasilinear framework, the alternative maximising total welfare is chosen when the number of voters is large (as shown by Eguia and Xefteris (2021)). However, outside the quasilinear framework, wealth effects may play a role: Wealthier voters will afford more votes. When wealth is distributed unequally and political preferences correlate with wealth, QV favours the rich.

In this paper, I analyse a variation of quadratic voting where voters pay for votes not with money but with an artificial currency ("vote credits") of which everyone is equally endowed.

*PhD candidate at Toulouse School of Economics. I thank my supervisor Karine van der Straeten for her advice and Fabien Gensbittel for his help with some of the mathematical aspects of this research.

To insulate election outcomes from the influence of wealth inequality, vote credits can only be spent on votes. This distinguishes it from QV with money, where private goods and votes are purchased with the same currency. To make quadratic voting meaningful, that is, to maintain some relationship between votes purchased and preference intensity, there must be several simultaneous polls on which the budget of vote credits can be spent. [1] I characterise voting equilibria as the number of voters tends to infinity and then contrast welfare outcomes between QV with vote credits and majority rule under different assumptions on the joint distribution of preferences. The results are mixed: Divorcing political power from wealth by introducing vote credits comes at the cost of giving more power to those with overall weak preferences across polls, i.e. a small opportunity cost of vote credits, and less to those with overall strong preferences, i.e. a high opportunity cost of vote credits. This problem arises as every voter faces the same budget constraint regardless of the overall preference intensity.[2] I show that, for large electorates, the sign of the vote total likely aligns with the sign of total utility when preferences are independent and identically distributed across the different polls and the number of simultaneous polls is large enough. These two assumptions (iid preferences across polls and many polls) serve to approximately equalise preference intensity across voters. In the opposite extreme, when preferences are perfectly correlated across polls, the outcome under QV with vote credits is the same as under majority rule whether or not the tyranny of the majority problem applies. The reason for this is that, when preferences are perfectly correlated, a voter with a strong preference on one issue always also has strong preferences on all other issues. This leaves no scope for expressing *relative* preference intensity so that QV with vote credits is reduced to a more count of supporters and opponents of the proposal at hand. Finally, I present a simple example in which Quadratic Voting does worse than majority voting on a utilitarian count. In this example, QV with vote credits favours a minority with arbitrarily weak preferences for a large enough electorate, while majority rule would pick the alternative that is favoured by the balance of preferences.

In recent years, QV has received international attention and different versions have been tried in different institutional contexts. Most famously, the Democratic Caucus of the Colorado House of Representatives used a version of quadratic voting with multiple alternatives to prioritise the allocation of funds to different projects. [3] QV has potential application outside of politics. Posner and Weyl (2014) make the case for Quadratic Voting (with money) as a tool of corporate governance. QV has also attracted the blockchain community's attention (e.g. Dimitri (2022)). An important shortcoming of QV may limit the range of settings in which it can successfully be applied: Both QV with money and with vote credits are in principle susceptible to collusion. Although collusion takes the form of a prisoners' dilemma in principle (defecting from any collusive agreement is a dominant strategy), the same can be said about logrolling (trade of votes across different polls under majority voting) and yet logrolling is an essential feature of parliamentary practice. It is unclear, how QV would per-

[1] When there is only one poll, every voter spends her entire budget on this poll irrespective of preference intensity as there is no other use for vote credits.

[2] To be precise, only relative preference intensities matter for a voter's allocation of votes and thus the mechanism accounts only for relative preferences.

[3] Recently, however a court ordered the end of the use of quadratic voting on grounds of lack of transparency as voting was secret.

form in the presence of collusion but the possibility of collusion casts doubt on the suitability of QV in an environment with both open ballots and repeated votes, i.e. parliaments. A more promising context for QV would be a direct democracy (with simultaneous referenda for the case of QV with vote credits) or large committees in organisations or companies, where voting is either secret, not repeated or a norm against collusion can be enforced.
As far as the theoretical literature on QV is concerned,Eguia and Xefteris (2021) analyse a binary vote-buying mechanism where votes are bought with money according to a large class of cost functions. As the number of voters approaches infinity, a quadratic cost function implements the alternative for which the total willingness to pay is larger. Eguia, Immorlica, et al. (2023) consider a context where one alternative out of several must be chosen and votes are again bought with money at a quadratic cost. As the number of voters approaches infinity, the probability that the welfare maximising alternative is chosen approaches 1. [4] Although QV with vote credits is plausibly a more attractive application in reality, it has not been theoretically analysed. This paper aims to fill this gap.

2 Model and Assumptions

A society consisting of N voters faces K simultaneous binary choices. For $k \in [K]$, denote by D_k the collective decision on choice k. D_k takes values in $\{1, -1\}$.

Types Each voter is characterised by a K-vector of preferences U_i, where U_{ik} denotes the additional utility voter i derives when $D_k = 1$ relative to $D_k = -1$. Thus, overall ex-post utility for voter i is given by $\sum_{k=1}^{K} \mathbb{1}\{D_k = 1\}U_{ik}$. Preferences U_i are drawn iid across voters from a continuous distribution F (see assumption 1). We denote by $\mathcal{U} := supp(F) \subseteq \mathbb{R}^K$ the type space, which we assume to be open and bounded. Let F_k denote the marginal distribution of preferences for choice k. Then $\mathcal{U}_k := supp(F_k) \subseteq \mathbb{R}$ is the set of possible preferences for choice k and we can write $\mathcal{U} = \bigtimes_{k=1}^{K} \mathcal{U}_k$. I also use the notation $\mathcal{U}_{-l} = \bigtimes_{k=1,k\neq l}^{K} \mathcal{U}_k$.

Assumption 1 *F is absolutely continuous with respect to the Lebesgue measure. Its support $\mathcal{U}$ is the product of bounded intervals $\mathcal{U}_1, ..., \mathcal{U}_K$.*

Actions To decide the K choices, K simultaneous polls are held. Each voter is endowed with a budget of 1 vote credit, which she uses to buy a number V_k of votes on each on poll. This number can be positive as well as negative. The cost of votes is quadratic in the number of votes purchased, i.e. the cost of v_k votes for poll k is v_k^2. To satisfy her budget constraint, each voter's set of possible actions is therefore limited to a vector of votes $v \in \mathbb{R}^K$ such that $\|v\|^2 = \sum_{k=1}^{K} v_k^2 \leq 1$. Thus, if we denote by V_i the random vector of votes of voter i, V_i takes values in the action set $\mathcal{A} := \{v \in \mathbb{R}^K : \|v\| \leq 1\}$.

[4] These results rely on the approximation that the marginal effect of a vote on the probability of choosing an alternative are identical across voters, which is exact only for an infinite number of voters. Moreover, both papers consider mechanisms in which the probability of choosing an alternative conditional on the vote total is a smooth function of the vote total instead of discontinuously switching from 0 to 1 as one alternative obtains the most votes. I am making use of the same approximations in the present paper.

Outcome and payoff The collective decision on each choice conditional on all the votes is stochastic and depends crucially on the average vote on each poll:

$$\mathbb{P}\left[D_k = 1 \middle| \frac{1}{N}\sum_{i=1}^{N} V_{ik} = x\right] = G(cx)$$

for some parameter $c > 0$, where $G \in \mathcal{C}^2$ is a smooth CDF-like function: G is strictly increasing from 0 to 1 such that $\lim_{x\to\infty} G(x) = 1$ and $\lim_{x\to-\infty} G(x) = 0$. Its derivative g is strictly positive everywhere and strictly decreasing in the absolute value of its argument. Its second derivative g' is bounded. c is a constant governing how fast the probability of making decision D_k switches from 0 to 1 as the average vote crosses the zero threshold. I will also refer to c as the sharpness of the mechanism. c could be arbitrarily large. Smoothing the social decision as a function of the average vote serves to make the problem mathematically tractable. To approximate outcomes under a mechanism where the social decision corresponds precisely to the sign of the vote total, I will let c tend to infinity in the propositions to follow.[5]

We can write the payoff conditional on type and actions of a voter of type $u \in \mathcal{U}$ when votes are $v_1, ..., v_N \in \mathbb{R}^K$ as:

$$\Pi(u, v_1, ..., v_N) := \sum_{k=1}^{K} u_k G\left(c\frac{1}{N}\sum_{i=1}^{N} v_{ik}\right)$$

Information and structure of the game Each voter knows only the realisation of her own preferences U_i. About other voters' preferences she knows only that they are drawn iid from distribution F. All voters simultaneously choose a vote vector in $\mathcal{A}$ (that is, they simultaneously choose how many votes to buy for each poll) before learning anything about other voters' actions.

Strategies and equilibrium concepts I denote the resulting game by Γ^N when there are N voters. Although the game also depends on other parameters, in particular c and F, which I will vary, I suppress them from the notation Γ^N and its equilibria to keep notation simple. The results will characterise symmetric Bayes Nash equilibria in pure strategies and later in mixed strategies. A pure strategy is a function $s : \mathcal{U} \to \mathcal{A}$ and a mixed strategy is a function $\sigma : \mathcal{U} \to \Delta(\mathcal{A})$. For now, whenever I use the term equilibrium, I am referring to a symmetric Bayes Nash equilibrium in pure strategies. I will identify such equilibria simply by their corresponding strategy s^N, i.e. when I refer to the equilibrium s^N, I am referring to an equilibrium whose strategy profile is $(s^N, s^N, ..., s^N)$.

[5]Note, however, that I do not prove that equilibria of the smooth game approximate equilibria of the discontinuous game as sharpness tends to infinity.

3 Key Results

3.1 Characterising the equilibrium

We start by establishing the existence of equilibria. For more detail, the reader is referred to the appendix.[6]

Proposition 1 *Under Assumption 1, a symmetric Bayes Nash Equilibrium in pure strategies exists.*

In the following, we consider sequences of games and sequences of equilibria in N. A sequence of equilibria $\{s^N\}_{N\in\mathbb{N}}$ of the sequence of games $\{\Gamma^N\}_{N\in\mathbb{N}}$ is a sequence of strategies such that each s^N constitutes an equilibrium in the N-player version of the voting game Γ^N. For some (or all) N, Γ^N may have multiple equilibria, in which case there are multiple possible sequences of equilibria. To characterise equilibria, we consider the optimisation problem of the voter. Suppose the last voter, voter N, has preferences $u \in \mathcal{U}$ and define $n := N-1$ the number of other voters. Voter N maximises expected utility without knowing other voters' preferences given that everyone else plays a pure strategy $s^N : \mathcal{U} \mapsto \mathcal{A}$,

$$\begin{aligned}\mathbb{E}\left[\sum_k u_k \mathbb{1}\{D_k = 1\}\right] &= \sum_k u_k \mathbb{E}\left[\mathbb{P}\left[D_k = 1 \middle| \frac{1}{N}\sum_{i=1}^{N} V_{ik} = \frac{1}{N}\left(\sum_{i=1}^{n} s^N(U_i) + v\right)\right]\right] \\ &= \sum_k u_k \mathbb{E}\left[G\left(c\frac{1}{N}\left(\sum_{i=1}^{n} s^N(U_i) + v\right)\right)\right],\end{aligned}$$

with respect to v. If s^N is a symmetric equilibrium it therefore satisfies

$$s^N(u) \in \arg\max_{v\in\mathcal{A}} \sum_{k=1}^{K} u_k \mathbb{E}\left[G\left(c\frac{1}{N}\left(\sum_{i=1}^{n} s_k^N(U_i) + v_k\right)\right)\right]$$

for all $u \in \mathcal{U}$. Since G is strictly increasing, this objective function is strictly increasing in any v_k such that $u_k > 0$ and strictly decreasing in any v_k such that $u_k < 0$. Therefore the budget constraint $\sum_{k=1}^{K} v_k^2 \leq 1$ is binding for any $u \in \mathcal{U}\backslash\{0\}$ and we obtain the following necessary optimality condition

$$s_k^N(u) = \frac{u_k \mathbb{E}\left[g\left(c\frac{1}{N}\left(\sum_{i=1}^{n} s_k^N(U_i) + s_k^N(u)\right)\right)\right]}{\sqrt{\sum_{l=1}^{K} u_l^2 \mathbb{E}\left[g\left(c\frac{1}{N}\left(\sum_{i=1}^{n} s_l^N(U_i) + s_l^N(u)\right)\right)\right]^2}}. \tag{1}$$

We simplify notation using the following definition:

[6] While I do not rule out the existence of asymmetric BNE, we can rule out symmetric BNE in which more than a measure zero set of types play mixed strategies (Lemma 7 in the appendix). There might be symmetric BNE in which a negligible set of types (that is, a set of types of measure equal to zero) play mixed strategies. But for each such equilibrium, the outcome is the same as in some truly pure symmetric BNE (Lemma 8 in the appendix). Therefore, we restrict attention from mixed to pure strategy symmetric BNE without loss .

Definition 1 *Given a sequence of equilibra* $\{s^N\}_{N\in\mathbb{N}}$*, the pivotality at* u *and approximate pivotality are defined as*

$$\tilde{p}_k^N(u) = \mathbb{E}\left[g\left(c\frac{1}{N}\left(\sum_{i=1}^{n} s_k^N(U_i) + s_k^N(u)\right)\right)\right] \quad \textit{and}$$
$$p_k^N = \mathbb{E}\left[g\left(c\frac{1}{N}\left(\sum_{i=1}^{n} s_k^N(U_i)\right)\right)\right]$$

for any $k \in [K]$*.*

We re-write condition 1 as

$$s_k^N(u) = \frac{u_k \tilde{p}_k^N(u)}{\sqrt{\sum_{l=1}^K u_l^2 \tilde{p}_l^N(u)^2}}.$$

The pivotalitiy at u, $\tilde{p}_k^N(u)$, captures the expected marginal impact of a vote on the probability of choosing $D_k = 1$. (The expected marginal impact is actually cp_k^N, but since only relative pivotalities across different polls matter, we can drop the constant c from the definition of pivotality without loss.) These pivotalities differ by type because each individual knows a different term of the average vote that determines the probability of implementing $D_k = 1$; each individual knows only their own vote. However, if the number of voters is large, the relative contribution of each individual voter is small and thus the values of $\tilde{p}_k^N(u)$ are close to each other for different types so that we can approximate it by the approximate pivotality p_k^N. The following proposition formalises this insight:

Proposition 2 *In any sequence of equilibria* $\{s^N\}_{N\in\mathbb{N}}$*,*

$$\lim_{N\to\infty} \sup_{u\in\mathcal{U}} |\tilde{p}_k^N(u) - p_k^N| = 0,$$

for all $k \in [K]$*.*

Proposition 2 together with optimality condition equation 1 suggest that, for large N, the functional form of the equilibrium s^N can be approximated by a simplified strategy t, parameterised by the vector $p^N := (p_1^N, ..., p_K^N)'$, i.e. $s_k^N(u) \approx t_k(u, p^N)$, where the simplified strategy $t : \mathcal{U} \times \mathbb{R}^K \mapsto \mathbb{R}^K$ is given by

$$t_k(u, q) := \frac{u_k q_k}{\sqrt{\sum_{l=1}^K u_l^2 {q_l}^2}}$$

for $u \in \mathcal{U}\backslash\{0\}$, $q \in \mathbb{R}^K\backslash\{0\}$ and $k \in [K]$ and $t_k(u, 0) = t_k(0, q) := 0$. (See Lemma 9 in the appendix.) How does p^N behave as N grows large? We can consider p_k^N as an inverse price of influence on choice k: The larger it is, the more influence a marginal unit of spending has on the choice probability and the more votes are allocated to it by individual voters. On the other hand, the more votes are allocated to a particular poll k, the lower is an individual

voter's influence on that poll, i.e. the smaller is p^N. This reasoning leads to the following approximate equilibrium condition:

$$p_k^N \approx \mathbb{E}\left[g\left(c\frac{1}{N}\sum_{i=1}^{N} t_k(U_i, p_k^N)\right)\right] \tag{2}$$

for all $k \in [K]$. This approximation is obtained simply by substituting the simplified strategy t_k for the actual strategy s_k^N in the definition of approximate pivotality. Let $\mathcal{P} := [\min_{x\in[-1,1]} g(x), \max_{x\in[-1,1]} g(x)]$ be the range of values that p_k^N can take. Defining the price mechanism $\psi : \mathcal{P}^K \mapsto \mathcal{P}^K$ by

$$\psi_k^N(q) = \mathbb{E}\left[g\left(c\frac{1}{N}\sum_{i=1}^{N} t_k(U_i, q)\right)\right]$$

for each $k \in [K]$, the approximate equilibrium condition 2 rewrites as

$$p_k^N \approx \psi_k^N(p^N) \tag{3}$$

for all $k \in [K]$. ψ^N is a sequence of functions which converges to a limit function

$$\psi := \lim_{N\to\infty} \psi^N = \left(g(c\mathbb{E}\left[t_k(U_i, \cdot)\right])\right)_{k\in[K]}$$

Definition 2 summarises these objects:

Definition 2 *The simplified strategy is a function $t : \mathcal{U} \times \mathbb{R}^K \mapsto \mathbb{R}^K$ given by*

$$t(u, q) := \left(\frac{u_k q_k}{\sqrt{\sum_{l=1}^{K} u_l^2 {q_l}^2}}\right)_{k\in[K]}$$

for all $u \in \mathcal{U}\{0\}$ and $q \in \mathbb{R}^K\backslash\{0\}$ and $t(u, 0) = t(0, q) := 0_K$. The range in which pivotality p_k^N is contained is $\mathcal{P} := [\min_{x\in[-1,1]} g(x), \max_{x\in[-1,1]} g(x)]$. The asymptotic price mechanism is $\psi : \mathcal{P}^K \mapsto \mathcal{P}^K$ with

$$\psi(q) = \left(g\left(c\mathbb{E}\left[t_k(U_i, q)\right]\right)\right)_{k\in[K]}$$

Let $\mathcal{Q}$ denote its set of fixed points.

Thus we can regard p^N as an approximate fixed point of ψ^N. As N approaches infinity, the equilibrium condition 3 becomes exact in the sense that p^N approaches the actual fixed points of the limit function ψ. The following proposition makes this more precise:

Proposition 3 *For any $a \in \mathbb{R}^K$ and any $\epsilon > 0$, define $B(a, \epsilon) = \{x \in \mathbb{R}^K | \|x - a\| < \epsilon\}$, the ϵ-neighborhood of a. Let $\mathcal{Q} = \{q \in \mathcal{P}^K : q = \psi(q)\}$ be the set of fixed points of ψ. For every $\epsilon > 0$ there is $N_\epsilon \in \mathbb{N}$ such that for all $N > N_\epsilon$*

$$p^N \in \bigcup_{q\in\mathcal{Q}} B(q, \epsilon).$$

(Note that Brouwer's Fixed Point Theorem guarantees the existence of fixed points of ψ.) Losely speaking, this proposition states that p^N is always close to some fixed point of ψ for large N, albeit not necessarily always the same one.
For the following welfare result we however require a more specific statement:

Conjecture 1 *For any fixed points q of ψ, there is a sequence of equilibria $\{s^N\}_{N\in\mathbb{N}}$ such that the associated sequence p^N converges to q.*

From this conjecture it would follow that for any fixed point q there is a sequence of equilibria $\{s^N\}_{N\in\mathbb{N}}$ converging (pointwise) to $t(\cdot, q)$.

3.2 Welfare properties

The simplified strategy, which approximates the true strategy as the number of voters tends to infinity, helps us understand the welfare properties of QV with vote credits. Suppose N is large and hence the pivotalities are approximately equal across types. Then the voting strategy for any poll $k \in [K]$ can be approximated by $s_k^N(u) \approx t_k(u, p^N) = u_k p_k^N \lambda^N(u)$, where $\lambda^N(u) = 1/\sqrt{\sum_{l=1}^K u_l^2 p_l^{N\,2}}$. The average vote on issue k is then equal to $p_k^N \frac{1}{N} \sum_{i=1}^N U_{ik} \lambda^N(U_i)$. Whether the sign of the vote total equals the sign of total welfare $\sum_{k=1}^K U_{ik}$, depends on how the weights $\lambda^N(U_i)$ relate to the U_{ik}. If these weights are constant or, more generally, uncorrelated with U_{ik}, the mechanism considered here is likely to implement the utilitarian social choice (at least as N tends to infinity and for a large c) as the sign on the vote total coincides with the sign of total welfare in this case. On the other hand, if weights are correlated to preferences (e.g. those with positive U_{ik} have smaller weights), it is not guaranteed that the signs of votal totals and total welfare coincide. What the weights capture is the opportunity cost of a vote. An individual with large preference intensities on choices where the equilibrium pivotality is high has a higher opportunity cost and therefore spends less of her budget on issue k for a given $u_k p_k^N$. This is reflected in a lower weight $\lambda^N(u)$. The basic problem is that a voter's budget does not adapt to her needs, i.e. to her overall preference intensity $\|U_i\|$.[7]
In the following we consider several extreme cases which allow us to make concrete welfare statements. First we consider conditions sufficient to restore the utilitarian guarantee obtained in the case of QV with money on a single issue when utility is quasi-linear. Suppose, that preferences are independent not only across voters but also across polls and the number of polls K is large. The large number of polls and the independence of preferences across polls reduces the variance of overall preference intensities $\|U_i\|$ across voters. This limits the variance of voters' opportunity costs. As a consequence, strategies are approximately linear in preferences for a large part of the distribution, restoring the desirable property that the signs of vote totals and of total utility coincide. We further assume that $\mu := \mathbb{E}[U_{ik}] \neq 0$ and that $\tau^2 := \mathbb{E}[U_{ik}^2] < \infty$.

Assumption 2 *$\{U_{ik}\} \overset{iid}{\sim} D$ across both i and k. Let $\mu = \mathbb{E}[U_{ik}] \neq 0$, $\tau^2 := \mathbb{E}[U_{ik}^2] < \infty$.*

[7] However, even if all voters had identical preference intensities, preferences are still interacted with pivotalities that may differ by issue giving rise to some variation in weights that would generally be correlated with preferences.

Note that in this case, the constant[8] vector $q^\star := (g(c\mathbb{E}[t_k(U_i, \iota_K)]))_{k\in[K]}$, where $\iota_K = (1, 1, ..., 1)' \in \mathbb{R}^K$, is a fixed point of ψ. Because $q^\star$ is constant and the simplified strategy t is homogeneous of degree zero in its second argument q, $t(u, q^\star) = t(u, \iota_K)$. Under conjecture 1, there is a sequence of equilibria $\{s^N\}_{N\in\mathbb{N}}$ converging to $t(u, \iota_K)$. In the particular mechanism treated here, the probability of choosing a particular alternative conditional on the vote total is not 0 or 1 but goes smoothly from 0 to 1 as a function of the average vote. Therefore, the mechanism does not necessarily make the decision that corresponds to the sign of the vote total. To be relatively sure that the mechanism makes the right decision, the sharpness constant c must also be very large. Accordingly, we let c tend to infinity in the following proposition. In words, the result states that for any probability π arbitrarily close to 1, there is c large enough such that the probability of making the utilitarian decision approaches or exceeds π as the number of voters tends to infinity.

Proposition 4 *Suppose assumptions 1 and 2 and conjecture 1 hold so that there is a sequence of equlibria $\{s^N\}_{N\in\mathbb{N}}$ with $\lim_{N\to\infty} s^N(u) = t(u, \iota_K)$ for all $u \in \mathcal{U}$. Then there is $\bar{K} \in \mathbb{N}$ such that, when the number of polls K exceeds $\bar{K}$,*

$$\lim_{c\to\infty} \lim_{N\to\infty} \mathbb{P}\left[D_k = \operatorname{sgn}\left(\sum_{i=1}^{N} U_{ik}\right)\right] = 1$$

for all $k \in [K]$ in any such sequence of equilibria.

Another extreme is the case of perfect correlation between preferences for different choices. Suppose the space of preferences is one-dimensional in the sense the every voters' type U_i is located on the same line (that is not orthogonal to any of the coordinate axes) as specified by the following assumption:

Assumption 3 *Preferences are given by $U_i = a_i b$, where $\{a_i\}$ are real-valued iid random variables such that $\mathbb{P}[a_i = 0] = 0$ and $\mathbb{E}[a_i] \neq 0$ and $b \in \mathbb{R}^K$ is a constant vector with only non-zero elements.*

Then voters with intense preferences on one poll relative to other voters also have relatively intense preferences on all other polls. (Using the notation of assumption 3, these are the voters with a large realisation of a_i.) But since their budget of vote credits does not adapt to the intensity of their preferences, they cannot express them with more votes. In a one-dimensional preference space, one can divide the electorate into two groups (corresponding to the signs of the a_i), within each of which there is complete agreement on all choices. The ratios of preferences for different polls are the same across all voters so that, given the linear homogeneity of the payoff function in type, vote choices are identical within groups. Moreover, as the number of voters tends to infinity, the vote vectors of one group approaches the negative of the vote vector of the other group. Thus, when N is large, Quadratic Voting amounts to a count of the number of members in each political camp - just like majority voting:

[8] It is constant because the $t_k(U_i, \iota_K)$ are identically distributed across $k \in [K]$ and thus their expectations are the same.

Proposition 5 *Let $V_{ik}^M \in \{-1, 1\}$ denote i's vote under majority voting. Under Assumption 3,*

$$\lim_{N\to\infty} \operatorname{sgn}\left(\sum_{i=1}^{N} V_{ik}\right) \overset{a.s.}{=} \lim_{N\to\infty} \operatorname{sgn}\left(\sum_{i=1}^{N} V_{ik}^M\right)$$

for all $k \in [K]$.

In both of the previous cases, Quadratic Voting with vote credits did at least as well as majority voting. Is QV with vote credits guaranteed to be as good or better than majority voting on the utilitarian criterion? The answer is no, as QV can give so much power to voters with overall weak preferences that it can choose the non-utilitarian outcome where majority voting would have chosen the utilitarian one. I demonstrate this with the following example: Suppose two simultaneous choices are voted on. On choice one, the preference distribution is symmetric about zero. The mean of preferences on choice two is positive. Suppose that voters with low preference intensities on choice one tend to have negative preferences on choice two while voters with intense preferences on either side of choice one tend to have positive preferences on choice two. Those who are indifferent on choice one will buy small amounts of votes on choice one and therefore exert outsized influence on choice two. Likewise, those who care highly about choice one will exert reduced influence on choice two. Hence it may be that those against choice two get their way despite the fact their negative utility is outweighed by the positive utility of those in favour. Those against choice two thus benefit from the conflict that those in favour of choice two fight over choice one. To formalise this, consider, in deviation from Assumption 1, a discrete type space with three types whose preferences on each of two polls are shown in the following table:

	Type 1	Type 2	Type 3
Poll 1	0	1	-1
Poll 2	$z < 0$	d	d

Table 1: Utility of different voter types on two simultaneous polls.

Suppose the probability that a voter is of type 1 is r and types 2 and 3 have equal probability. Whenever $z > d\frac{r-1}{r}$, it is optimal from a utilitarian perspective to choose $D_2 = 1$. When the type space is discrete, a symmetric equilibrium still exists although not necessarily in pure strategies:

Proposition 6 *Suppose $\mathcal{U}$ is a finite subset of $\mathbb{R}^K$. A symmetric Bayes Nash Equilibrium (possibly in mixed strategies) exists.*

For the rest of the section, the term "equilibrium" can refer to any symmetric Bayes Nash Equilibrium, not only pure strategy ones. When strategies are mixed, the probability that a player plays an action that is a best response to the other players' mixed strategy is equal to 1. Thus, with probability 1, V_i satisfies the same optimality condition as s^N in the case of a pure strategy equilibrium,

$$V_{ik} = \frac{u_{ik}\tilde{p}_k^N(V_{ik})}{\sqrt{\sum_{k=1}^K u_{ik}\tilde{p}_k^N(V_{ik})}}, \tag{4}$$

except with a different definition of pivotality:

Definition 3 *When the type space is finite, we define the pivotality at vote v for issue k corresponding to the symmetric mixed strategy equilibrium σ for the N-player game Γ^N*

$$\tilde{p}_k^N(v) = \mathbb{E}_\sigma\left[g\left(c\frac{1}{N}\left(\sum_{i=1}^n V_{ik} + v\right)\right)\right]$$

and the approximate pivotality is given by

$$p_k^N = \mathbb{E}_\sigma\left[g\left(c\frac{1}{N}\sum_{i=1}^n V_{ik}\right)\right].$$

When the equilibrium is not pure, the action played is no longer a function of the type so that we have to write it as a function of the action actually chosen. Nonetheless, the remaining asymptotic results still hold. In particular,

Proposition 7 *In any sequence of mixed strategy equilibria*

$$\lim_{N\to\infty}\sup_{v\in[-1,1]}|\tilde{p}_k^N(v) - p_k^N| = 0$$

for all $k \in [K]$.

Thus, the approximation of actual voting with a simplified strategy $t(\cdot, p_k^N)$ is still valid (for more detail, see Lemma 15 in the appendix), which means that the range in which actions played in a mixed strategy equilibrium are located collapses towards $t(\cdot, p_k^N)$ as N tends to infinity. As a consequence, Proposition 3 still goes through so that we can draw inference on the set of voting equilibria from the set of fixed points of ψ. This leads to the following proposition:

Proposition 8 *Suppose the type space and distribution is as described by Table 1. For any $c > 0$, for any $N \in \mathbb{N}$, for any equilibrium σ of Γ^N, let $\rho_\sigma^N := \mathbb{E}_\sigma\left[G\left(c\frac{1}{N}\sum_{i=1}^N V_{i2}\right)\right]$ denote the probability that $D_2 = 1$ is chosen in poll 2. If $-d^2 + d\sqrt{d^2+1} < 0.5$ and $r \in (-d^2 + d\sqrt{d^2+1}, 0.5)$, then for any $c > 0$ there is a convergent sequence $\{X^N\}_{N\in\mathbb{N}}$ of upper bounds such that in any sequence of equilibria $\{\sigma^N\}$:*

$$\rho_{\sigma^N}^N \leq X^N$$

and

$$\lim_{c\to\infty}\lim_{N\to\infty} X^N = 0.$$

Proposition 9 tells us that as the voting mechanism becomes arbitrarily sharp ($c \to \infty$), and the number of voters is large ($N \to \infty$), the probability that $D_2 = 1$ is chosen is close to zero in any equilibrium. In particular, these results are entirely independent of type 1's preference on poll 2, z, as long as it is negative. (This is trivial, as type 1 always votes $(0, -1)$ for any $z < 0$.) This means that QV may choose $D_2 = -1$ even when z is very small and the balance of preferences overwhelmingly favours $D_2 = 1$.

4 Literature

In this paper, I am comparing QV with vote credits to majority rule where voters vote sincerely. Sincere voting is a dominant strategy under majority rule when voters cannot trade votes with other voters. In many situations, however, logrolling, the trade of votes across different polls, is a realistic possibility allowing voters to express preference intensity even under majority rule: Suppose voter X is strongly in favour of proposal 1 and weakly opposed to proposal 2 while voter Y is weakly opposed to proposal 1 and strongly in favour of proposal 2. They may exchange X's vote on proposal 2 for Y's vote on proposal 1 and thus each vote in favour of both proposals. If the votes are pivotal on both issues, both voters gain from this trade. However, general welfare effects of vote trades are difficult to analyse under majority rule with logrolling (see Casella and Macé (2021)). The logic of private goods markets, where trade is pareto-improving, does not apply to exchanges of votes because unlike exchanges of goods, exchanges of votes also affect all other voters if the votes exchanged are pivotal. Further, one trade between a pair or groups of voters can motivate further trades between other voters. When at a given allocation of votes, no coalition of voters finds it beneficial to trade, the allocation of voters is said to be stable. Riker and Brams (1973) focus on pairwise trades and provides an example of a sequence of trades which leads to a stable allocation of votes at which the voting outcome is pareto-inferior to the outcome under majority rule without trading. On the other hand, when we allow for trade between any coalition of voters (instead of restricting it to pairwise trades), it is easily seen that any stable allocation of votes must produce a pareto-efficient outcome: Any allocation producing an inefficient outcome would be blocked by the coalition of the whole electorate, which could organise a reallocation leading to an efficient outcome. Casella and Palfrey (2019) show also that sequences of trades do not generally lead to the Condorcet winner. Positive welfare results akin to proposition 4 are not available in the literature.

My approach of assuming a large number of simultaneous choices, across which preferences are independent is strongly reminiscent of Jackson and Sonnenschein (2007). In their contribution, a number of individuals need to make collective decisions on a number K of identical problems. An individual has an independently drawn preference type for each problem (this would correspond to the U_{ik} rather than U_i in the present paper) and the distribution of types is known by the social planner. The paper considers a mechanism where individuals report their preferences for each issue under the constraint that the frequencies of preferences reported must (closely) match the preference distribution. This forces individuals to "lie" on a fraction of problems but for a large number of problems, this fraction is small. This need to lie is analogous to the constraint that all voters must report a vote vector of norm smaller than 1 regardless of the overall size of their preferences in the QV with vote credits mechanism analysed in this paper. As in Jackson and Sonnenschein, a large number of simultaneous choices limits the extent of inaccurate reporting buy rendering large deviations of average preference intensity across choices from expected preference intensity unlikely. Jackson and Sonnenschein's mechanism can implement any ex-ante pareto-efficient social choice function as the number of problems approaches infinity. An extended version of the mechanism is additionally robust to collusion. Like the original mechanism, it makes use of knowledge of the true distribution. However, as neither mechanism relies one a large number of individuals, the information elicited by the mechanism does add value over the

social planner's knowledge of the distribution.[9]
The principle that a cost of voting elicits voters preference intensities also bears on the results from the costly voting literature. This literature is mainly concerned with majority rule but introduces a cost of participating in the election. Krishna and Morgan (2015) study majority voting with an random number of eligible voters whose voting costs are independent of their political preferences. They show that, as long as the probability of having a voting cost equal to zero is zero, majority rule leads to the utilitarian choice with probability approaching one as the number of voters tends to infinity. If this shift from an equilibrium where the majority's preferred option is always chosen when there are no costs to an equilibrium where mean utility is maximised regardless of how small the variance of the cost distribution is seems puzzling, the key is to recognise that this welfare result is an asymptotic result: The smaller the variance of the cost distribution, the larger the number of voters needed to bring the pivotalities down to an order of magnitude comparable to the cost scale. While the alternatives to be voted on are exogenous in Krishna and Morgan, Ledyard (1984) studies a model with two candidates choosing platforms to maximise their probability of winning given an electorate with voting costs. When the distribution of costs starts at zero, both candidates choose the platform maximising the mean utility of voters with zero voting costs.

5 Discussion

This paper studies QV with vote credits using a model where the social choice probability is a smooth function of the average vote to approximate a game where the decision deterministically corresponds to the sign of the vote balance. It remains to be proven that the limit of equilibria as the sharpness of the decision function approaches infinity equals the equilibrium of the true QV voting game.
A possible extension for future research is a QV mechanism where several alternatives are available on each poll (i.e. an extension of Eguia, Immorlica, et al. (2023) to several simultaneous polls and vote credits instead of money). Another interesting extension would be to embed QV in a model with continuous policy dimensions and study the political equilibria under QV.[10]

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A Proofs

A.1 Existence of Equilibrium

To prove the existence of symmetric pure Bayes Nash equilibria, I largely follow Eguia and Xefteris (2021) with some modifications: I first prove the existence of a symmetric Bayes Nash equilibrium in mixed strategies and then show that in any such mixed strategy equilibrium almost all types play a pure strategy. A mixed strategy σ is a function from the set of types $\mathcal{U}$ to the set of distributions on the action space $\mathcal{A}$. We denote by $\mathcal{M}$ the set of mixed strategies.

To prove the existence of symmetric Bayes Nash equilibria, following Eguia and Xefteris (2021), we additionally make use of ex-ante Bayes Nash Equilibria and of distributional strategies.

Definition 4 (Ex-ante Bayes Nash Equilibrium) *An ex-ante Bayes Nash equilibrium is a profile of mixed strategies $\sigma \in \mathcal{M}^N$, one for each player, such that, before learning her type, any player is best responding to all other players. That is, for every $i \in [N]$,*

$$\sigma_i \in \arg\max_{x \in \mathcal{M}} \sum_{k=1}^{K} \mathbb{E}_{(\sigma_1,...,\sigma_{i-1},x,\sigma_{i+1},...,\sigma_N)} \left[\mathbb{1}\{D_k = 1\} U_{ik}\right].$$

(Here, for $\gamma \in \mathcal{M}^N$, $\mathbb{E}_\gamma$ is the expectation resulting from players playing strategy profiles $\gamma_1, ..., \gamma_N$.)

To distinguish ex-ante Bayes Nash equilibria from standard Bayes Nash equilibria, we call the latter *interim* Bayes Nash equilibria. The difference between the set of of interim Bayes Nash equilibria and ex-ante Bayes-Nash equilibria is that the latter also includes strategy profiles where players play strategies that are suboptimal for a measure zero set of types.

Definition 5 (Distributional strategy) *Following Milgrom and Weber (1985), a distributional strategy μ is a distribution on the combinations of types and actions $\mathcal{U} \times \mathcal{A}$ such that its marginal distribution of types equals the actual distribution of types F. Let $\mathcal{D}$ denote the set of all distributional strategies.*

The set of distributional strategies is compact under the topology of weak convergence. Compactness of distributional strategies allows us to prove existence of a symmetric equilibrium in distributional strategies. Inferring the existence of a symmetric interim BNE in pure strategies of a derivative game $\hat{\Gamma}^N$ whose pure strategies are the distributional strategies of the original game Γ^N, we then reason our way to the existence of a symmetric interim BNE in mixed strategies via ex-ante BNE and from there to symmetric interim BNE in pure strategies.

Assumption 1 *F is absolutely continuous with respect to the Lebesgue measure. Its support $\mathcal{U}$ is the product of bounded intervals $\mathcal{U}_1, ..., \mathcal{U}_K$.*

Lemma 1 *Under Assumption 1, the set of distributional strategy profiles $\mathcal{D}^N$ of Γ^N is compact in the weak topology.*

Proof. $\mathcal{D}$ is compact in the topology of weak convergence (see for example Milgrom and Weber (1985), page 626). The Cartesian product of two compact sets is compact. Thus by induction, the set of profiles of distributional strategies $\mathcal{D}^N$ is compact for any number of players $N \in \mathbb{N}$. ■

Lemma 2 (Glicksberg, 1952) *Let S be a convex compact subset of a convex Hausdorff linear topological vector space. Let $\Omega : S \rightrightarrows S$ be a convex-valued correspondence whose graph is closed. Ω has a fixed point $x \in \Omega(x)$.*

Proof. See Glicksberg (1952).

Lemma 3 *Under Assumptions 1, Γ^N has a symmetric Bayes Nash equilibrium in distributional strategies for all $N \in \mathbb{N}$.*

Proof. Glicksberg (1952) uses lemma 2 to show that games with compact Hausdorff pure strategy spaces and continuous payoff functions have equilibria. We will follow along the same lines to show that a *symmetric* equilibrium in distributional strategies exists.
Consider the game $\hat{\Gamma}^N$ with N players whose pure strategies are the distributional strategies of the voting game Γ^N and whose payoffs are the corresponding expected payoffs

$$\hat{\Pi}_i : \mathcal{D}^N \mapsto \mathbb{R}, \quad \hat{\Pi}_i(\mu_1, ..., \mu_N) := \mathbb{E}_{\mu_i}\left[\mathbb{E}_{\mu_1 \times \mu_{i-1} \times \mu_{i+1} ... \times \mu_N}\left[\Pi(U_i, V_1, ..., V_N)|U_i, V_i\right]\right]$$

We show that the best response correspondence of one player to all other players playing the same strategy $\mu \in \mathcal{D}$ of this game,

$$BR_i : \mathcal{D} \rightrightarrows \mathcal{D}, \quad BR_i(\mu) = \arg\max_{\lambda \in \mathcal{D}} \hat{\Pi}_i(\mu, ..., \mu, \lambda, \mu, ...\mu),$$

satisfies the conditions of lemma 2. First, as already mentioned in the proof of Lemma 1, $\mathcal{D}$ is compact and, trivially, it is convex. Second, convexity of $BR(\mu)$ follows from the linearity of the expectation operator in one distribution. Further, the payoff Π is bounded function ($|\Pi(u, v_1, ..., v_n)| \leq \sum_{i=1}^{K} |u_k|$ for all $u \in U$ and $v_1, ..., v_N \in \mathcal{A}$ and $\mathcal{U}$ is bounded) and continuous (because G is continuous). Therefore, by definition of the topology of weak convergence, $\hat{\Pi}_i$ is continuous. Thus, $\hat{\Pi}_i$ is a continuous function on a compact set so $\max_\lambda \hat{\Pi}_i(\mu, ..., \mu, \lambda, \mu, ...\mu)$ exists. Thus, $BR_i(\mu)$ is the inverse image of the closed set $\{\max_\lambda \hat{\Pi}_i(\mu, ..., \mu, \lambda, \mu, ...\mu)\}$ under a continuous function and therefore closed. Further, as its domain is a compact topological vector space, it is closed. From Berge's Maximum Theorem we know that BR_i is upper-hemicontinuous.[11] An upper-hemicontinuous closed-valued correspondence with a closed domain has a closed graph. Therefore, BR_i satisfies the conditions of 2 and therefore has at least one fixed point $x \in BR_i(x)$. Since the functions $\hat{\Pi}_1, ..., \hat{\Pi}_N$ are symmetric, $BR_1, ..., BR_N$ are identical and have the same fixed points. Thus, any of these fixed points is a symmetric Nash equilibrium in pure strategies of $\hat{\Gamma}^N$ and any such equilibrium is a symmetric Bayes Nash equilibrium in distributional strategies of Γ^N. ■

Lemma 4 *For any symmetric Bayes Nash equilibrium in distributional strategies of* Γ^N *there is a corresponding symmetric interim Bayes Nash equilibrium in mixed strategies.*

Proof. Given a symmetric Bayes Nash equilibrium in the distributional strategy μ, the corresponding mixed strategy σ is such that for any measurable subset of the action space $a \subseteq \mathcal{A}$, and any type $u \in \mathcal{U}$, $\sigma(u)(a) = \mu(\{u\} \times a)$ (*actually* $\mu(\{u\} \times a) = 0$. *how to say this correctly?* The profile σ generally corresponds to an ex-ante Bayes Nash equilibrium rather than an interim Bayes Nash equilibrium since any distributional interim BNE strategy can assign suboptimal behaviour to a measure zero set of types. Let $\tilde{\mathcal{U}}$ be this subset of not best responding types under σ. Let the mixed strategy σ' be a strategy where every type $u \in \mathcal{U}$ is best responding to a situation where all other voters play σ. σ' is a symmetric interim Bayes Nash equilibrium since it differs from σ only for a measure zero set of types, so that for any strategy, any voter has the same expected payoff when all other voters play σ' as

[11] Berge's Maximum Theorem applies as $\hat{P}i_i$ is continuous, there is no constraint on which strategies a player can choose as a function of the strategy of the other players and the strategy set is compact.

when all other voters play σ. ■

The proof that a measure 1 set of types plays pure strategies also follows Eguia and Xefteris (2021) closely:

Definition 6 *The interim expected payoff function is defined as:*

$$\bar{\Pi} : \mathcal{U} \times A \times \mathcal{M}^{N-1} \mapsto \mathbb{R}, \quad \bar{\Pi}(u, v, \xi) = \mathbb{E}_{\xi} \left[\Pi(u, V_1, ..., V_{i-1}, v, V_{i+1}, ...V_N) \right]$$

The best response correspondence is defined as:

$$BR : \mathcal{U} \times \mathcal{M}^{N-1} \rightrightarrows \mathcal{A}, \quad BR(u, \xi) = \arg\max_{v \in \mathcal{A}} \bar{\Pi}(u, v, \xi)$$

Projections of best responses onto kth coordinate axis:

$$BR_k(u, \xi) := \{a \in [-1, 1] : \exists b \in BR(u, \xi) : b_k = a\}$$

Lemma 5 *The coordinate projections of the best response set $BR_k(u, \xi)$ has a maximum and a minimum for each issue $k \in [K]$.*

Proof. The function $f : x \in A \mapsto \bar{\Pi}(u, x, \xi)$ is a continuous function on a compact set $\mathcal{A}$ for any $(u, \xi) \in \mathcal{U} \times \mathcal{M}$. Therefore it has maxima, i.e. $BR(u, \xi)$ is non-empty for any $(u, \xi) \in \mathcal{U} \times \mathcal{M}$. Further, $BR(u, \xi) = \arg\max_x \bar{\Pi}(x, \xi)$, the inverse image of the closed set $\{\max_x \bar{\Pi}(x, \xi)\}$, is closed, bounded and therefore compact. The projection $BR_k(u, \xi)$ is the image of the compact set $BR(u, \xi)$ under the continuous function $g_k : (x_1, ..., x_K) \in \mathbb{R}^K \mapsto x_k \in \mathbb{R}$ and therefore it is compact and attains its supremum and infimum. ■

Definition 7

$$\overline{BR}_k(u, \xi) = \max\{BR_k(u, \xi)\} \quad \textit{and} \quad \underline{BR}_k(u, \xi) = \min\{BR_k(u, \xi)\}$$

Lemma 6 *Let $u, u' \in \mathcal{U}$ and $k \in [K]$ be such that $u'_k > u_k$ and $u'_l = u_l$ for all $l \neq k$. Then $\underline{BR}_k(u', \xi) \geq \overline{BR}_k(u, \xi)$ for any $\xi \in \mathcal{M}^{N-1}$.*

Proof. For arbitrary $\xi \in \mathcal{M}^{N-1}$, let $b \in BR(u, \xi)$ be a best response such that $b_k = \overline{BR}_k(u, \xi)$. For any $v \in A$:

$$\bar{\Pi}(u, b, \xi) - \bar{\Pi}(u, v, \xi) \geq 0$$

i.e.

$$\begin{aligned} u_k \left(\mathbb{P}_{\xi} \left[D_k = 1 | V_{ik} = b_k \right] - \mathbb{P}_{\xi} \left[D_k = 1 | V_{ik} = v_k \right] \right) + \\ \sum_{l \in [K] \setminus \{k\}} u_l \left(\mathbb{P}_{\xi} \left[D_l = 1 | V_{il} = b_l \right] - \mathbb{P}_{\xi} \left[D_l = 1 | V_{il} = v_l \right] \right) \geq 0 \end{aligned} \tag{5}$$

Suppose v is such that $v_k < b_k$. Because G is strictly increasing, the probability that k is implemented is strictly larger when i buys b_k votes than when she buys v_k votes on issue k: $\mathbb{P}_{\xi} \left[D_k = 1 | V_{ik} = b_k) \right] - \mathbb{P}_{\xi} \left[D_k = 1 | V_{ik} = v_k \right] > 0$. Now consider u', which differs from u only in the kth coordinate, where it is strictly larger and compute $\bar{\Pi}(u', b, \xi) - \bar{\Pi}(u', v, \xi)$. When

replacing u by u' in inequality 5, only the first term changes; it becomes larger. Thus, the difference in interim expected payoffs between playing b and playing v for an individual with preference u' is strictly positive:

$$\bar{\Pi}(u', b, \xi) - \bar{\Pi}(u', v, \xi) > 0$$

This means that any v with $v_k < b_k$ cannot be a best response for a voter with preference u', i.e. $\underline{BR}_k(u', \xi) \geq \overline{BR}_k(u, \xi)$ for any $\xi \in \mathcal{M}^{N-1}$. ■

Lemma 7 *Under Assumption 1,* $\mathbb{P}[|BR(U, \xi)| = 1] = 1$ *for all* $\xi \in \mathcal{M}^{N-1}$.

Proof. For each $k \in [K]$, each $x \in \mathcal{U}_{-k}$ and $\xi \in \mathcal{M}^{N-1}$ consider the functions

$$\overline{br}_k^{x,\xi} := \overline{BR}_k(x_1, ..., x_{k-1}, \cdot, x_k, ..., x_{K-1}, \xi)$$

and

$$\underline{br}_k^{x,\xi} := \underline{BR}_k(x_1, ..., x_{k-1}, \cdot, x_k, ..., x_{K-1}, \xi)$$

which are increasing by Lemma 6. We first argue that $BR(u, \xi)$ must be single-valued for any (u, ξ) such that all functions $\left\{\overline{br}_k^{u_{-k},\xi}\right\}_{k\in[K]}$ (where u_{-k} is obtained from u by omitting its kth coordinate) are continuous at their respective u_k: Suppose that at $(u, \xi) \in \mathcal{U} \times \mathcal{M}^{N-1}$, $BR(u, \xi)$ is not single-valued. Then there is $k \in [K]$ such that $\overline{BR}_k(u, \xi) > \underline{BR}_k(u, \xi)$, i.e. $\overline{br}_k^{u_{-k},\xi}(u_k) > \underline{br}_k^{u_{-k},\xi}(u_k)$. For one such k, let $\epsilon = \overline{br}_k^{u_{-k},\xi}(u_k) - \underline{br}_k^{u_{-k},\xi}(u_k)$. By Lemma 6, $\overline{br}_k^{u_{-k},\xi}(x) \leq \underline{br}_k^{u_{-k},\xi}(y)$ for all $x < y$. Since $\overline{br}_k^{u_{-k},\xi}$ is increasing, $\overline{br}_k^{u_{-k},\xi}(u_k) - \overline{br}_k^{u_{-k},\xi}(x) \geq \overline{br}_k^{u_{-k},\xi}(u_k) - \underline{br}_k^{u_{-k},\xi}(u_k) = \epsilon > 0$ for any $x < u_k$ so that $\overline{br}_k^{u_{-k},\xi}$ is discontinuous at u_k. Secondly, we show that the set of $u \in \mathcal{U}$ at which $\overline{br}_k^{u_{-k},\xi}$ has a discontinuity for some k has Lebesgue measure 0: For $k \in [K]$, $x \in \mathcal{U}_{-k}$ and $\xi \in \mathcal{M}^{N-1}$, let $d_k(x, \xi)$ denote the set of discontinuity points $x \in \mathcal{U}_k$ of $\overline{br}_k^{u_{-k},\xi}$. Since $\overline{br}_k^{x,\xi}$ is monotone, $d_k(x, \xi)$ is countable and therefore it has Lebesgue measure 0: $L^1(d_k(x, \xi)) = 0$ for all $x \in \mathcal{U}_{-k}$. Therefore the Lebesgue measure of discontinuities in the direction of the kth coordinate across *all* slices of $\mathcal{U}$ is 0: Define the set $D_k(\xi) := \{u \in \mathcal{U} | u_k \in d_k(u_{-k}, \xi)\}$. Its measure is:

$$\begin{aligned}
L^K(D_k(\xi)) &= \int_{\mathcal{U}} \mathbb{1}\{x \in D_k(\xi)\}\, dL^K(x) \\
&= \int_{\mathcal{U}_{-k}} \int_{\mathcal{U}_k} \mathbb{1}\{x_k \in d_k(x_{-k}, \xi)\}\, dL^1(x_k)\, dL^{K-1}(x_{-k}) \\
&= \int_{\mathcal{U}_{-k}} L^1(d_k(x_{-k}, \xi))\, dL^{K-1}(x_{-k}) \\
&= \int_{\mathcal{U}_{-k}} 0\, dL^{K-1}(x_{-k}) \\
&= 0
\end{aligned}$$

Therefore the set $D(\xi) := \bigcup_{k\in[K]} D_k(\xi)$ of points in $\mathcal{U}$ at which some $\overline{br}_k^{u_{-k},\xi}$ has a discontinuity for some $k \in [K]$ is a countable union of measure 0 sets and has measure 0. Let $M(\xi) = \{u \in \mathcal{U} | |BR(u, \xi)| > 1\}$ be the set of points at which there is more than one

best response. From the first part of this proof we remember that $M(\xi) \subseteq D(\xi)$. Thus $L^K(M(\xi)) = 0$ and because the distribution of preferences F is absolutely continuous with respect to the Lebesgue measure:

$$\mathbb{P}\left[|BR(u,\xi)| > 1\right] = 0$$

■

Definition 8 *For a symmetric mixed strategy profile $(\sigma, \sigma, ..., \sigma) \in \mathcal{M}^N$, define the set of best responding pure strategies*

$$\mathcal{S}(\sigma, \sigma, ..., \sigma) = \{s : \mathcal{U} \mapsto \mathbb{R}^K \,|\, \forall u \in \mathcal{U} :\ s(u) \in BR(u, \sigma, \sigma, ..., \sigma)\}$$

Lemma 8 *For any $(\sigma, \sigma, ..., \sigma) \in \mathcal{M}^N$, $\mathcal{S}(\sigma, \sigma, ..., \sigma)$ is non-empty. If mixed strategy $\sigma \in \mathcal{M}$ constitutes a symmetric Bayes Nash Equilibrium, then any $s \in \mathcal{S}(\sigma, \sigma, ..., \sigma)$ constitutes a pure symmetric Bayes Nash Equilibrium.*

Proof. $\mathcal{S}(\sigma)$ is non-empty as for any $(u, \xi) \in \mathcal{U} \times \mathcal{M}$, $BR(u, \xi)$ is non-empty (see the proof of Lemma 5). Since $s(u) \in BR(u, \sigma, ...)$, it follows from Lemma 7 that a player's action playing $s \in \mathcal{S}(\sigma, \sigma, ..., \sigma)$ and playing σ coincides with probability 1. Thus $BR(u, s, s, ...) = \mathcal{S}(\sigma, \sigma, ..., \sigma)$ and hence $s(u) \in BR(u, s, s, ...)$ for all $u \in \mathcal{U}$, i.e. s is a symmetric BNE. ■

Proposition 1 *Under Assumption 1, a symmetric Bayes Nash Equilibrium in pure strategies exists.*

Proof. This follows by combining the statements of Lemmas 3, 4 and 8.

A.2 Asymptotics in N

Restating Definition 1 for convenience.

Definition 1 *Given a sequence of equilibra $\{s^N\}_{N\in\mathbb{N}}$, the pivotality at u and approximate pivotality are defined as*

$$\begin{aligned}\tilde{p}_k^N(u) =& \mathbb{E}\left[g\left(c\frac{1}{N}\left(\sum_{i=1}^{n} s_k^N(U_i) + s_k^N(u)\right)\right)\right] \quad \textit{and} \\ p_k^N =& \mathbb{E}\left[g\left(c\frac{1}{N}\left(\sum_{i=1}^{n} s_k^N(U_i)\right)\right)\right]\end{aligned}$$

for any $k \in [K]$.

Proposition 2 *In any sequence of equilibria $\{s^N\}_{N\in\mathbb{N}}$,*

$$\lim_{N\to\infty} \sup_{u\in\mathcal{U}} |\tilde{p}_k^N(u) - p_k^N| = 0,$$

for all $k \in [K]$.

Proof. Differentiability of g allows us to apply the Mean Value Theorem and write:

$$\mathbb{E}\left[g\left(c\frac{1}{N}\left(\sum_{i=1}^{n}s_k^N(U_i)+s_k^N(u)\right)\right)\right] =\mathbb{E}\left[g\left(c\frac{1}{N}\sum_{i=1}^{n}s_k^N(U_i)\right)\right]+ c\frac{1}{N}s_k^n(u)\mathbb{E}\left[g'\left(c\left(\sum_{i=1}^{n}s_k^N(U_i)\right)/N+\delta\right)\right]$$

for some $\delta \in (\min\{\sum_{i=1}^n s_k^N(U_i), \sum_{i=1}^n s_k^N(U_i)+s_k^N(u)\}/N, \max\{\sum_{i=1}^n s_k^N(U_i), \sum_{i=1}^n s_k^N(U_i)+s_k^N(u)\}/N)$. Since $|s_k^N(u)| \leq 1$,

$$\left|\mathbb{E}\left[g\left(c\frac{1}{N}\left(\sum_{i=1}^{n}s_k^N(U_i)+s_k^N(u)\right)\right)\right]-\mathbb{E}\left[g\left(c\frac{1}{N}\sum_{i=1}^{n}s_k^N(U_i)/N\right)\right]\right| \leq c\frac{1}{N}\sup_{x\in[-1,1]}g'(x)$$

for all $u \in \mathcal{U}$, $N \in \mathbb{N}$ and $k \in [K]$. ■

Definition 2 *The simplified strategy is a function $t : \mathcal{U} \times \mathbb{R}^K \mapsto \mathbb{R}^K$ given by*

$$t(u,q) := \left(\frac{u_k q_k}{\sqrt{\sum_{l=1}^K u_l^2 {q_l}^2}}\right)_{k\in[K]}$$

for all $u \in \mathcal{U}\{0\}$ and $q \in \mathbb{R}^K\backslash\{0\}$ and $t(u,0) = t(0,q) := 0_K$. The range in which pivotality p_k^N is contained is $\mathcal{P} := [\min_{x\in[-1,1]} g(x), \max_{x\in[-1,1]} g(x)]$. The asymptotic price mechanism is $\psi : \mathcal{P}^K \mapsto \mathcal{P}^K$ with

$$\psi(q) = \left(g\left(c\mathbb{E}\left[t_k(U_i,q)\right]\right)\right)_{k\in[K]}$$

Let $\mathcal{Q}$ denote its set of fixed points.

Lemma 9 *For all $u \in \mathcal{U}$ and $k \in [K]$: $\lim_{N\to\infty} s_k^N(u) - t_k(u,p^N) = 0$.*

Proof. If s^N is an equilibrium in pure strategies it satisfies

$$s^N(u) \in \arg\max_{v\in\mathcal{A}} \sum_{k=1}^{K} u_k \mathbb{E}\left[G\left(c\frac{1}{N}\left(\sum_{i=1}^{n}s_k^N(U_i)+v_k\right)\right)\right]$$

for all $u \in \mathcal{U}$. Since G is strictly increasing, this objective function is strictly increasing in any v_k such that $u_k > 0$ and strictly decreasing in any v_k such that $u_k < 0$. Therefore the constraint $\sum_{k=1}^K v_k^2 \leq 1$ is binding for any $u \in \mathcal{U}\backslash\{0\}$. Since G is continuous, this objective function is continuous. Therefore, the following optimality conditions provide a necessary condition on $s^N(u)$ for all $u \in \mathcal{U}\backslash\{0\}$

$$\begin{aligned} 2s_k^N(u)\lambda(u) =& u_k c\frac{1}{N}\mathbb{E}\left[g\left(c\frac{1}{N}\sum_{i=1}^{n}s_k^N(U_i)+v_k\right)\right] \\ \sum_{l=1}^{K} s_l^N(u)^2 =& 1 \end{aligned}$$

for all $k \in [K]$ and some $\lambda(u) > 0$. Combining these equations yields the necessary condition

$$s_k^N(u) = \frac{u_k \mathbb{E}\left[g\left(c\frac{1}{N}\left(\sum_{i=1}^N s_k^N(U_i) + s_k^N(u)\right)\right)\right]}{\sqrt{\sum_{l=1}^K u_l^2 \mathbb{E}\left[g\left(c\frac{1}{N}\left(\sum_{i=1}^N s_l^N(U_i) + s_l^N(u)\right)\right)\right]^2}}$$

We can write $s_k^N(u) = t_k(u, \tilde{p}^N(u))$. $t_k(u, \cdot)$ is continuous and hence uniformly continuous on the compact set $\mathcal{P} := [\min_{x\in[-1,1]} g(x), \max_{x\in[-1,1]} g(x)]$. By Proposition 2, $\lim_{n\to\infty} \tilde{p}_k^N(u) - p_k^N = 0$, so $\forall \epsilon > 0$ there is N_ϵ such that $|\tilde{p}_k^N(u) - p_k^N| < \epsilon$ for all $N > N_\epsilon$ and both p_k^N and $\tilde{p}_k^N(u)$ are in $\mathcal{P}$ for all $N \in \mathbb{N}$. By uniform continuity of $t_k(u, \cdot)$, for any $\delta > 0$ there is $\epsilon_\delta > 0$ such that if $|\tilde{p}_k^N(u) - p_k^N| < \epsilon$, $|s_k^N(u) - t_k(u, p^N)| = |s(u, \tilde{p}^N(u)) - t_k(u, p^N)| < \delta$. So for any $\delta > 0$, for all $N > N_{\epsilon_\delta}$, $|s(u, \tilde{p}^N(u)) - t(u, p^N)| < \delta$, which establishes the statement of the lemma. ■

Lemma 10

$$\lim_{N\to\infty} \|p^N - \psi^N(p^N)\| = 0$$

Proof. g is continuously differentiable so the Mean Value Theorem is applicable to it:

$$\begin{aligned} p_k^N =& \mathbb{E}\left[g\left(c\frac{1}{N}\left(\sum_{i=1}^n t_k(U_i, p^N) + s_k^N(U_i) - t_k(U_i, p^N)\right)\right)\right] \\ =& \mathbb{E}\left[g\left(c\frac{1}{N}\left(\sum_{i=1}^n t_k(U_i, p^N)\right)\right)\right] + \\ & \mathbb{E}\left[c\frac{1}{N}\sum_{i=1}^n \left(s_k^N(U_i) - t_k(U_i, p^N)\right) g'\left(c\frac{1}{N}\sum_{i=1}^n t_k(U_i, p^N) + \delta^N\right)\right] \end{aligned} \tag{6}$$

for some random sequence $|\delta^N|$. We note that the sum in the second expectation is the sum of $n(= N-1)$ identically distributed terms and apply the Cauchy-Schwarz inequality to bound the second term of this sum by something that converges to 0:

$$\left|\mathbb{E}\left[c\frac{1}{N}\sum_{i=1}^N \left(s_k^N(U_i) - t_k(U_i, p^N)\right) g'\left(c\frac{1}{N}\sum_{i=1}^N t_k(U_i, p^N) + \delta^N\right)\right]\right| \leq$$
$$c\frac{n}{N}\sqrt{\mathbb{E}\left[\left(s_k^N(U_i) - t_k(U_i, p^N)\right)^2\right] \mathbb{E}\left[g'\left(c\frac{1}{N}\sum_{i=1}^N t_k(U_i, p^N) + \delta^N\right)^2\right]}$$

The second term of this upper bound is bounded since g' is bounded. We show that the first term converges to 0: $s_k^N(u) - t_k(u, p^N) \to 0$ for all $u \in \mathcal{U}$ as we know from Lemma 9 and $|s_k^N(u) - t_k(u, p^N)| \leq 1$. Thus, by the Dominated Convergence Theorem:

$$\begin{aligned} \lim_{N\to\infty} \mathbb{E}\left[\left(s_k^N(U_i) - t_k(U_i, p^N)\right)^2\right] =& \mathbb{E}\left[\lim_{N\to\infty} \left(s_k^N(U_i) - t_k(U_i, p^N)\right)^2\right] \\ =& 0 \end{aligned}$$

The first term of equation 6, $\mathbb{E}\left[g\left(c\frac{1}{N}\sum_{i=1}^{N} t_k(U_i, p^N)\right)\right]$, is equal to $\psi_k^N(p^N)$. Therefore $\lim_{N\to\infty} |p_k^N - \psi_k^N(p^N)| = 0$ for all $k \in [K]$, which establishes the statement of the lemma. ■

Lemma 11 *ψ^N converges uniformly to ψ on $\mathcal{P}^K$.*

Proof. For any $q \in \mathcal{P}^K$, $\frac{1}{N}\sum_{i=1}^{N} t_k(U_i, q) \overset{\text{a.s.}}{\to} \mathbb{E}\left[t_k(U_i, q)\right]$ by the Strong Law of Large Numbers. g is continuous and therefore $g\left(\frac{1}{N}\sum_{i=1}^{N} t_k(U_i, q)\right) \overset{\text{a.s.}}{\to} g\left(\mathbb{E}\left[t_k(U_i, q)\right]\right)$ by the Continuous Mapping Theorem. g is bounded so that we can apply the Dominated Convergence Theorem to infer pointwise convergence:

$$\begin{aligned}
\lim_{N\to\infty} \psi_k^N(q) &= \lim_{N\to\infty} \mathbb{E}\left[g\left(c\frac{1}{N}\sum_{i=1}^{N} t_k(U_i, q)\right)\right] \\
&= \mathbb{E}\left[\lim_{N\to\infty} g\left(c\frac{1}{N}\sum_{i=1}^{N} t_k(U_i, q)\right)\right] \\
&= g\left(c\mathbb{E}\left[t_k(U_i, q)\right]\right) \\
&= \psi_k(q)
\end{aligned}$$

This hold for each k in $[K]$ so that we can write $\lim_{N\to\infty} \psi^N(q) = \psi(q)$. For any $u \in \mathcal{U}$, $t_k(u, \cdot)$ is continuous and thus ψ^N is continuous for all $N \in \mathbb{N}$. (This can again be shown using the Dominated Convergence Theorem.) Therefore, given compactness of $\mathcal{P}^K$, ψ^N converges uniformly to ψ on $\mathcal{P}^K$. ■

Lemma 12 *ψ has at least one fixed point.*

Proof. By the same argument as in Lemma 11, ψ is continuous and, in addition to being compact, $\mathcal{P}^K$ is also convex. The statement therefore follows from Brouwer's Fixed Point Theorem. ■

Proposition 3 *For any $a \in \mathbb{R}^K$ and any $\epsilon > 0$, define $B(a, \epsilon) = \{x \in \mathbb{R}^K | \|x - a\| < \epsilon\}$, the ϵ-neighborhood of a. Let $\mathcal{Q} = \{q \in \mathcal{P}^K : q = \psi(q)\}$ be the set of fixed points of ψ. For every $\epsilon > 0$ there is $N_\epsilon \in \mathbb{N}$ such that for all $N > N_\epsilon$*

$$p^N \in \bigcup_{q\in\mathcal{Q}} B(q, \epsilon).$$

Proof.

$$\|p^N - \psi(p^N)\| \leq \|p^N - \psi^N(p^N)\| + \|\psi^N(p^N) - \psi(p^N)\|$$

The first term of this upper bound converges to 0 by Lemma 10 and the the second term converges to 0 because p^N takes only values on $\mathcal{P}^K$ and ψ^N converges uniformly to ψ on $\mathcal{P}^K$ by Lemma 11. Thus $\lim_{N\to\infty} \|p^N - \psi(p^N)\| = 0$.
Let $\mathcal{Q}$ be the set of fixed points of ψ. For any $a \in \mathbb{R}^K$ and any $\epsilon > 0$, define $B(a, \epsilon) = \{x \in \mathbb{R}^K | \|x - a\| < \epsilon\}$. We prove by contradiction that for any $\epsilon > 0$ there is $N_\epsilon \in \mathbb{N}$ such that for all $N > N_\epsilon$, there is a fixed point $q \in \mathcal{Q}$ (not necessarily the same fixed point for all

$N > N_\epsilon$) such that $\|p^N - q\| < \epsilon$. Suppose not. Then there is $\epsilon > 0$ such that for all $\tilde{N} \in \mathbb{N}$ there is $N > \tilde{N}$ such that $\|p^N - q\| \geq \epsilon$ for all $q \in \mathcal{Q}$, i.e. $p^N \in \mathcal{P}^K \setminus \bigcup_{q\in\mathcal{Q}} B(q,\epsilon)$. Therefore

$$\|p^N - \psi(p^N)\| \geq \inf_{x \in \mathcal{P}^K \setminus \bigcup_{q\in\mathcal{Q}} B(q,\epsilon)} \|x - \psi(x)\| \tag{7}$$

for this ϵ and N.
Since $\bigcup_{q\in\mathcal{Q}} B(q,\epsilon)$ is a union of open sets, it is open, therefore $\mathcal{P}^K \setminus \bigcup_{q\in\mathcal{Q}} B(q,\epsilon)$ is closed, bounded and hence compact. $x \mapsto \|x - \psi(x)\|$ is continuous because ψ is continuous (by a similar argument as for the continuity of ψ^N in Lemma 11). Hence $x \mapsto \|x - \psi(x)\|$ attains its infimum on $\mathcal{P}^K \setminus \bigcup_{q\in\mathcal{Q}} B(q,\epsilon)$, i.e.

$$\inf_{x \in \mathcal{P}^K \setminus \bigcup_{q\in\mathcal{Q}} B(q,\epsilon)} \|x - \psi(x)\| = \min_{x \in \mathcal{P}^K \setminus \bigcup_{q\in\mathcal{Q}} B(q,\epsilon)} \|x - \psi(x)\|$$

But $\min_{x \in \mathcal{P}^K \setminus \bigcup_{q\in\mathcal{Q}} B(q,\epsilon)} \|x - \psi(x)\| > 0$ because $\mathcal{P}^K \setminus \bigcup_{q\in\mathcal{Q}} B(q,\epsilon)$ does not contain any fixed points of ψ. Then inequality 7 implies that $\|p^N - \psi(p^N)\|$ does not converge to 0, which contradicts the finding we made at the start of this proof. ■

A.3 Welfare

Lemma 13 *Suppose* $\lim_{N\to\infty} s^N(u) = t(u, \iota_K)$ *for all* $u \in \mathcal{U}$. *Then for all* K,

$$\frac{1}{N}\sum_{i=1}^{N} s^N(U_i) \xrightarrow{p} \mathbb{E}\left[t(U_i, \iota_K)\right]$$

Proof.

$$\begin{aligned}\frac{1}{N}\sum_{i=1}^{N} s^N(U_i) =& \frac{1}{N}\sum_{i=1}^{N} \left(t(U_i,\iota_K) + s^N(U_i) - t(U_i,\iota_K)\right) \\ =& \frac{1}{N}\sum_{i=1}^{N} t(U_i,\iota_K) + \frac{1}{N}\sum_{i=1}^{N} \left(s^N(U_i) - t(U_i,\iota_K)\right)\end{aligned} \tag{8}$$

The law of large numbers is applicable to the first term, i.e. $\frac{1}{N}\sum_{i=1}^{N} t(U_i,\iota_K) \xrightarrow{p} \mathbb{E}\left[t(U_i,\iota_K)\right]$. The second term converges to 0 in mean:

$$\begin{aligned}\mathbb{E}\left[\left\|\frac{1}{N}\sum_{i=1}^{N}\left(s^N(U_i) - t(U_i,\iota_K)\right)\right\|\right] &\leq \mathbb{E}\left[\frac{1}{N}\sum_{i=1}^{N}\left\|s^N(U_i) - t(U_i,\iota_K)\right\|\right] \\ &= \mathbb{E}\left[\|s^N(U_i) - t(U_i,\iota_K)\|\right] \\ &\overset{N\to\infty}{\to} 0\end{aligned}$$

To derive this convergence we can apply the Dominated Convergence Theorem as in the proof of Lemma 10 since $\|s^N(U_i) - t(U_i,\iota_K)\|$ is bounded (by 2) and converges to 0 pointwise. Convergence in mean implies convergence in probability, i.e. $\left\|\frac{1}{N}\sum_{i=1}^{N} s^N(U_i) - t(U_i,\iota_K)\right\| \xrightarrow{p} 0$.
Thus, applying Slutsky Theorem to equation 8, $\frac{1}{N}\sum_{i=1}^{N} s^N(U_i) \xrightarrow{p} \mathbb{E}\left[t(U_i,\iota_K)\right]$. ■

Assumption 2 *$\{U_{ik}\} \overset{iid}{\sim} D$ across both i and k. Let $\mu = \mathbb{E}[U_{ik}] \neq 0$, $\tau^2 := \mathbb{E}[U_{ik}^2] < \infty$.*

Lemma 14 *Under assumptions 2, there is $\bar{K}$ such that for all $K > \bar{K}$, $\operatorname{sgn}(\mathbb{E}[t_k(U_i, \iota_K]) = \operatorname{sgn}(\mu)$.*

Proof. We will prove this by proving $\lim_{K\to\infty} \mathbb{E}\left[\sqrt{K} t_k(U_i, \iota_K)\right] = \mu/\sqrt{\tau^2}$. For some $M > 0$ we define the iid set of bounded random variables $U_{ikM}^2 := \min\{U_{ik}^2, M\}$. Let ϵ be a real number in $(0, \mathbb{E}[U_{ikM}^2])$. Therefore, $\epsilon < \tau^2$. We decompose $\mathbb{E}[t_k(U_i, \iota_K)]$ as follows:

$$\begin{aligned}\mathbb{E}\left[\sqrt{K} t_k(U_i, \iota_K)\right] =& \mathbb{E}\left[\sqrt{K} t_k(U_i, \iota_K) \mathbb{1}\left\{\sum_{l=1}^{K} U_{ik}^2/K \geq \epsilon\right\}\right] + \\ & \mathbb{E}\left[\sqrt{K} t_k(U_i, \iota_K) \mathbb{1}\left\{\sum_{l=1}^{K} U_{ik}^2/K < \epsilon\right\}\right]\end{aligned} \tag{9}$$

The first expectation is the expectation of sequence of random variables which are uniformly dominated by an integrable random variable

$$t_k(U_i, \iota_K) \mathbb{1}\left\{\sum_{l=1}^{K} U_{ik}^2/K \geq \epsilon\right\} = \frac{U_{ik}}{\sqrt{\sum_{k=1}^{K} U_{il}^2/K}} \mathbb{1}\left\{\sum_{l=1}^{K} U_{ik}^2/K \geq \epsilon\right\} \leq U_{ik}/\sqrt{\epsilon}.$$

Moreover, the strong law of large numbers is applicable to $\sum_{k=1}^{K} U_{il}^2/K$ since $\mathbb{E}[U_{ik}^2] = \tau^2$ is finite by assumption. Hence $\sum_{k=1}^{K} U_{il}^2/K \overset{a.s.}{\to} \tau^2$ and $\mathbb{1}\left\{\sum_{l=1}^{K} U_{ik}^2/K \geq \epsilon\right\} \overset{a.s.}{\to} 1$ because $\epsilon < \tau^2$. It follows from Slutsky Theorem that

$$\frac{U_{ik}}{\sqrt{\sum_{k=1}^{K} U_{il}^2/K}} \mathbb{1}\left\{\sum_{l=1}^{K} U_{ik}^2/K \geq \epsilon\right\} \overset{a.s.}{\to} U_{ik}/\tau^2.$$

Thus we can apply Lebesgue's Dominated Convergence Theorem to the sequence in K

$$\mathbb{E}\left[\sqrt{K} t_k(U_i, \iota_K) \mathbb{1}\left\{\sum_{l=1}^{K} U_{ik}^2/K \geq \epsilon\right\}\right] :$$

$$\begin{aligned}\lim_{K\to\infty} \mathbb{E}\left[\sqrt{K} t_k(U_i, \iota_K) \mathbb{1}\left\{\sum_{l=1}^{K} U_{ik}^2/K \geq \epsilon\right\}\right] =& \mathbb{E}\left[\lim_{K\to\infty} \sqrt{K} t_k(U_i, \iota_K) \mathbb{1}\left\{\sum_{l=1}^{K} U_{ik}^2/K \geq \epsilon\right\}\right] \\ =& \mu/\tau^2\end{aligned}$$

Now we need only show that the second expectation in 9 vanishes, which we will do based on Hoeffding's inequality. First we bound it above using the fact the $t_k(U_i, \iota_K) \leq 1$:

$$\mathbb{E}\left[\sqrt{K} t_k(U_i, \iota_K) \mathbb{1}\left\{\sum_{l=1}^{K} U_{ik}^2/K < \epsilon\right\}\right] \leq \sqrt{K} \mathbb{P}\left[\sum_{l=1}^{K} U_{ik}^2/K < \epsilon\right] \tag{10}$$

Since $\sum_{l=1}^{K} U_{ilM}^2 \leq \sum_{l=1}^{K} U_{il}^2$, $\mathbb{P}\left[\sum_{l=1}^{K} U_{il}^2 < \epsilon\right] \leq \mathbb{P}\left[\sum_{l=1}^{K} U_{ilM}^2 < \epsilon\right]$. The latter probability decays exponentially in K:

$$\begin{aligned}\mathbb{P}\left[\sum_{l=1}^{K} U_{ilM}^2/K < \epsilon\right] =& \mathbb{P}\left[\sum_{l=1}^{K} U_{ik}^2 - K\tau^2 < K(\epsilon - \tau^2)\right] \\ <& \exp\left(-2\frac{K^2(\tau^2-\epsilon)^2}{KM^2}\right) \\ =& \exp\left(-2\frac{K(\tau^2-\epsilon)^2}{M^2}\right)\end{aligned}$$

This inequality is an application of Hoeffding's inequality the sum of bounded random variables $\sum_{l=1}^{K} U_{ikM}^2$. This upper bound dominates the $\sqrt{K}$-term in equation 10 showing that the second term of equation 9 converges to zero. Thus

$$\begin{aligned}\lim_{K\to\infty} \mathbb{E}\left[\sqrt{K} t_k(U_i, \iota_K)\right] =& \lim_{K\to\infty} \mathbb{E}\left[\sqrt{K} t_k(U_i, \iota_K) \mathbb{1}\left\{\sum_{l=1}^{K} U_{ik}^2/K \geq \epsilon\right\}\right] \\ =& \mu/\tau^2\end{aligned}$$

As a consequence, there is $\bar{K}$ such that for all $K > \bar{K}$, $\operatorname{sgn}\left(\mathbb{E}\left[t_k(U_i, \iota_K)\right]\right) \equiv \operatorname{sgn}\left(\mathbb{E}\left[\sqrt{K} t_k(U_i, \iota_K)\right]\right) = \operatorname{sgn}(\mu)$. ■

Proposition 4 *Suppose assumptions 1 and 2 and conjecture 1 hold so that there is a sequence of equlibria $\{s^N\}_{N\in\mathbb{N}}$ with $\lim_{N\to\infty} s^N(u) = t(u, \iota_K)$ for all $u \in \mathcal{U}$. Then there is $\bar{K} \in \mathbb{N}$ such that, when the number of polls K exceeds $\bar{K}$,*

$$\lim_{c\to\infty} \lim_{N\to\infty} \mathbb{P}\left[D_k = \operatorname{sgn}\left(\sum_{i=1}^{N} U_{ik}\right)\right] = 1$$

for all $k \in [K]$ in any such sequence of equilibria.

Proof. We know from the proof of Lemma 13 that $\frac{1}{N}\sum_{i=1}^{N} s_k^N(U_i) \xrightarrow{p} \mathbb{E}\left[t_k(U, \iota_K)\right]$. For any $\epsilon > 0$, define the random variable $I_\epsilon := \mathbb{1}\{|\frac{1}{N}\sum_{i=1}^{N} s_k^N(U_i) - \mathbb{E}\left[t_k(U_i, \iota_K)\right]| < \epsilon\}$. Without loss of generality, suppose $\mu > 0$. For $k \in [K]$,

$$\begin{aligned}\mathbb{P}\left[D_k = \operatorname{sgn}(\mu)\right] =& \mathbb{E}\left[G\left(c\frac{1}{N}\sum_{i=1}^{N} s_k^N(U_i)\right)\right] \\ =& \mathbb{E}\left[G\left(c\frac{1}{N}\sum_{i=1}^{N} s_k^N(U_i)\right)(I_\epsilon + (1 - I_\epsilon))\right]\end{aligned} \tag{11}$$

We note that for any $\epsilon > 0$,

$$\lim_{N\to\infty} \mathbb{E}\left[G\left(c\frac{1}{N}\sum_{i=1}^{N} s_k^N(U_i)\right)(1 - I_\epsilon)\right] = 0$$

because G is bounded and $\frac{1}{N}\sum_{i=1}^{N} s_k^N(U_i) \xrightarrow{p} \mathbb{E}\left[t_k(U, \iota_K)\right]$. (This is shown using the Cauchy-Schwarz inequality as done in some previous lemmas). Therefore

$$\lim_{N\to\infty} \mathbb{E}\left[G\left(c\frac{1}{N}\sum_{i=1}^{N} s_k^N(U_i)\right)\right] - \mathbb{E}\left[G\left(c\frac{1}{N}\sum_{i=1}^{N} s_k^N(U_i)\right) I_\epsilon\right] = 0 \tag{12}$$

As a consequence, for any $\epsilon > 0$

$$\liminf_{N\to\infty} \mathbb{E}\left[G\left(c\frac{1}{N}\sum_{i=1}^{N} s_k^N(U_i)\right)\right] = \liminf_{N\to\infty} \mathbb{E}\left[G\left(c\frac{1}{N}\sum_{i=1}^{N} s_k^N(U_i)\right) I_\epsilon\right] \text{ and} \tag{13}$$

$$\limsup_{N\to\infty} \mathbb{E}\left[G\left(c\frac{1}{N}\sum_{i=1}^{N} s_k^N(U_i)\right)\right] = \limsup_{N\to\infty} \mathbb{E}\left[G\left(c\frac{1}{N}\sum_{i=1}^{N} s_k^N(U_i)\right) I_\epsilon\right] \tag{14}$$

Further, for any $\epsilon > 0$ and all $N \in \mathbb{N}$,

$$G\left(c\left(\mathbb{E}\left[t_k(U, \iota_K)\right] - \epsilon\right)\right)\mathbb{E}\left[I_\epsilon\right] \leq \mathbb{E}\left[G\left(c\frac{1}{N}\sum_{i=1}^{N} s_k^N(U_i)\right) I_\epsilon\right] \leq G\left(c\left(\mathbb{E}\left[t_k(U, \iota_K)\right] + \epsilon\right)\right)\mathbb{E}\left[I_\epsilon\right].$$

Given that $\lim_{N\to\infty} \mathbb{E}\left[I_\epsilon\right] = 1$, this means that

$$\liminf_{N\to\infty} \mathbb{E}\left[G\left(c\frac{1}{N}\sum_{i=1}^{N} s_k^N(U_i)\right) I_\epsilon\right] \geq G\left(c\left(\mathbb{E}\left[t_k(U, \iota_K)\right] - \epsilon\right)\right) \text{ and} \tag{15}$$

$$\limsup_{N\to\infty} \mathbb{E}\left[G\left(c\frac{1}{N}\sum_{i=1}^{N} s_k^N(U_i)\right) I_\epsilon\right] \leq G\left(c\left(\mathbb{E}\left[t_k(U, \iota_K)\right] + \epsilon\right)\right). \tag{16}$$

Given equations 13 and 14 and the fact that inequalities 15 and 16 hold for any $\epsilon > 0$, we can conclude

$$\begin{aligned}
&\liminf_{N\to\infty} \mathbb{E}\left[G\left(c\frac{1}{N}\sum_{i=1}^{N} s_k^N(U_i)\right)\right] \\
=&\limsup_{N\to\infty} \mathbb{E}\left[G\left(c\frac{1}{N}\sum_{i=1}^{N} s_k^N(U_i)\right)\right] \\
=&\lim_{N\to\infty} \mathbb{E}\left[G\left(c\frac{1}{N}\sum_{i=1}^{N} s_k^N(U_i)\right)\right] \\
=&G\left(c\mathbb{E}\left[t_k(U_i, \iota_K)\right]\right)
\end{aligned}$$

Under the assumption that $\operatorname{sgn}\left(\mathbb{E}\left[s_k(U_i, \iota_K)\right]\right) = \operatorname{sgn}\left(\mu\right)$ for large enough K and since $\mathbb{E}\left[t_k(U_i, \iota_K)\right]$ does not depend on c, this means

$$\begin{aligned}
&\lim_{c\to\infty}\lim_{N\to\infty} \mathbb{P}\left[D_k = \operatorname{sgn}\left(\mu\right)\right] \\
=&\lim_{c\to\infty}\lim_{N\to\infty} \mathbb{E}\left[G\left(c\frac{1}{N}\sum_{i=1}^{N} s_k^N(U_i)\right)\right] \\
=&\lim_{c\to\infty} G\left(c\mathbb{E}\left[t_k(U_i, \iota_K)\right]\right) \\
=&1
\end{aligned}$$

(The last equality follows from the assumption on G that $\lim_{x\to\infty} G(x) = 1$). When $\text{sgn}(\mu) = -1$, symmetric reasoning applies given the assumption $\lim_{x\to-\infty} G(x) = 0$. ■

Assumption 3 *Preferences are given by $U_i = a_i b$, where $\{a_i\}$ are real-valued iid random variables such that $\mathbb{P}[a_i = 0] = 0$ and $\mathbb{E}[a_i] \neq 0$ and $b \in \mathbb{R}^K$ is a constant vector with only non-zero elements.*

Proposition 5 *Let $V_{ik}^M \in \{-1, 1\}$ denote i's vote under majority voting. Under Assumption 3,*

$$\lim_{N\to\infty} \text{sgn}\left(\sum_{i=1}^{N} V_{ik}\right) \overset{a.s.}{=} \lim_{N\to\infty} \text{sgn}\left(\sum_{i=1}^{N} V_{ik}^M\right)$$

for all $k \in [K]$.

Proof . Firstly, we show that for any $N \in \mathbb{N}$, s^N is homogeneous of degree 0 for positive factors. For a particular distribution of other voters' votes $V_1, ..., V_n$, suppose $s(u)$ maximises the expected utility of voter N when she is of type u, i.e. $\mathbb{E}[\Pi(u, V_1, ..., V_n, s(u))] \geq \mathbb{E}[\Pi(u, V_1, ..., V_n, v)]$ for all $v \in \mathcal{A}$. Expected utility is homogeneous (of degree one) in type:

$$\begin{aligned}
\mathbb{E}[\Pi(\lambda u, V_1, ..., V_n, v)] =& \sum_{k=1}^{K} \lambda u_k \mathbb{E}\left[G\left(c\frac{1}{N}\left(\sum_{i=1}^{n} V_{ik} + v_k\right)\right)\right] \\
=& \lambda \sum_{k=1}^{K} u_k \mathbb{E}\left[G\left(c\frac{1}{N}\left(\sum_{i=1}^{n} V_{ik} + v_k\right)\right)\right] \\
=& \lambda \mathbb{E}[\Pi(u, V_1, ..., V_n, v)]
\end{aligned}$$

for any $\lambda \in \mathbb{R}$, any $u \in \mathcal{U}$ and any $v \in \mathcal{A}$. Thus for any $\lambda > 0$, $\mathbb{E}[\Pi(\lambda u, V_1, ..., V_n, s(u)] = \lambda\mathbb{E}[\Pi(u, V_1, ..., V_n, s(u)))] \geq \lambda\mathbb{E}[\Pi(u, V_1, ..., V_n, v)] = \mathbb{E}[\Pi(\lambda u, V_1, ..., V_n, v)]$ for any $v \in \mathcal{A}$. the equilibrium strategy s^N maximises expected utility when all other voters also play s^N. Thus it is homogeneous of degree zero for positive scalars and we can write: $s_k^N(a_i b) = s_k^N(b)$ if $a_i > 0$ and $s_k^N(a_i b) = s_k^N(-b)$ if $a_i < 0$, for all k. Thus we can write the equilibrium average vote as

$$\frac{1}{N}\sum_{i=1}^{N} V_{ik} \overset{a.s.}{=} \frac{N_+}{N} s_k^N(b) + \frac{N_-}{N} s_k^N(-b),$$

where $N_+ = \sum_{i=1}^{N} \mathbb{1}\{a_i > 0\}$ and $N_- = \sum_{i=1}^{N} \mathbb{1}\{a_i < 0\}$ are the counts of voters on either side of all decisions, respectively. The equality is almost sure because $\mathbb{P}[a_i = 0] = 0$ by assumption. We know from Lemma 9 that $\lim_{N\to\infty} \|s^N(u) - t(u, p^N)\| = 0$ for any $u \in \mathcal{U}$. The function t is homogeneous of degree zero in its first argument for both positive and negative scalars, more precisely $t(\lambda u, q) = \text{sgn}(\lambda)\, t(u, q)$ for any $u \in U$, $q \in \mathcal{P}^K$ and $\lambda \in \mathbb{R}$. Therefore

$$s_k^N(b) + s_k^N(-b) = s_k^N(b) - t_k(b, p^N) + s_k^N(-b) - t(-b, p^N) \overset{N\to\infty}{\to} 0$$

and we can write $s_k^N(-b) = -s_k^N(b) + \epsilon_N$ for some sequence $\epsilon_N \to 0$. Thus the average vote can be written as

$$\frac{1}{N}\sum_{i=1}^{N} V_{ik} = \frac{N_+ - N_-}{N} s_k^N(b) + \frac{N_-}{N}\epsilon_N \tag{17}$$

By the strong law of large numbers $\frac{N_+}{N} \overset{a.s.}{\to} \mathbb{P}[a_i > 0]$ and $\frac{N_-}{N} \overset{a.s.}{\to} \mathbb{P}[a_i < 0]$. Moreover, we know that $s_k^N(b)$ cannot converge to zero: Since $b \neq 0$, $\|s^N(b)\| = 1$. Therefore, for any N there is a poll $l(N) \in [K]$ such that $s_{l(N)}^N(b)^2 \geq 1/K$. From the necessary condition 1 derived from the voter's optimisation problem in Lemma 9, we know that for any two polls $k, l \in [K]$:

$$s_k^N(b) = \frac{b_k}{b_l}\frac{\tilde{p}_k^N(b)}{\tilde{p}_l^N(b)} s_l^N(b)$$

Since the pivotalities are expectations of the bounded function g evaluated at the average vote, the ratio of pivotalities $\frac{\tilde{p}_k^N(b)}{\tilde{p}_l^N(b)} \in \left[\frac{min_{x\in[-1,1]}g(cx)}{max_{x\in[-1,1]}g(cx)}, \frac{max_{x\in[-1,1]}g(cx)}{min_{x\in[-1,1]}g(cx)}\right]$. Thus

$$|s_k^N(b)| \geq |b_k/b_{l(N)}|\frac{min_{x\in[-1,1]}g(cx)}{max_{x\in[-1,1]}g(cx)}1/\sqrt{K} \geq |b_k/b_{max}|\frac{min_{x\in[-1,1]}g(cx)}{max_{x\in[-1,1]}g(cx)}1/\sqrt{K},$$

where b_{max} is the largest element of b.[12] As a consequence, on the events that $N_+/N \to \mathbb{P}[a_i > 0]$ and $N_-/N \to \mathbb{P}[a_i < 0]$ and remembering that $\epsilon_N \to 0$,

$$\operatorname{sgn}\left(\liminf_{N\to\infty} \frac{1}{N}\sum_{i=1}^{N} V_{ik}\right) = \operatorname{sgn}\left(\mathbb{P}[a_i > 0] - \mathbb{P}[a_i < 0]\right)\operatorname{sgn}(b_k).$$

Hence with probability one, there is a $\bar{N} \in \mathbb{N}$ such that

$$\operatorname{sgn}\left(\frac{1}{N}\sum_{i=1}^{N} V_{ik}\right) = \operatorname{sgn}\left(\mathbb{P}[a_i > 0] - \mathbb{P}[a_i < 0]\right)\operatorname{sgn}(b_k)$$

for all $N > \bar{N}$, in other words

$$\lim_{N\to\infty} \operatorname{sgn}\left(\frac{1}{N}\sum_{i=1}^{N} V_{ik}\right) \overset{a.s.}{=} \operatorname{sgn}\left(\mathbb{P}[a_i > 0] - \mathbb{P}[a_i < 0]\right)\operatorname{sgn}(b_k).$$

Under majority voting (where a vote in favour counts as $+1$ and a vote against counts as -1), the average vote is given by $\frac{1}{N}\sum V_{ik}^M = (N_+ - N_-)/N\operatorname{sgn}(b_k)$ so again

$$\lim_{N\to\infty} \operatorname{sgn}\left(\frac{1}{N}\sum_{i=1}^{N} V_{ik}^M\right) \overset{a.s.}{=} \operatorname{sgn}\left(\mathbb{P}[a_i > 0] - \mathbb{P}[a_i < 0]\right)\operatorname{sgn}(b_k).$$

The statement of the proposition follows. ■

[12] This lower bound is very close to 0 for large c so that it seems that N must be large in order for the probability of the majority vote and the quadratic vote to coincide with high probability. But with some additional work, the bounds on the ratio of pivotalities could be much improved by noting that when few votes are cast on an issue, the pivotality on that issue is actually high.

A.4 Discrete type space

Proposition 6 *Suppose $\mathcal{U}$ is a finite subset of $\mathbb{R}^K$. A symmetric Bayes Nash Equilibrium (possibly in mixed strategies) exists.*

Proof. A pure strategy is a function $\mathcal{U} \mapsto \mathcal{A}$. When the type space $\mathcal{U}$ is finite with $|\mathcal{U}|$ elements, it can be characterised by a matrix $s \in \mathbb{R}^{K\times|\mathcal{U}|}$ such that each column has norm at most 1 (to satisfy the budget constraint for each type realisation). Under the norm $a \in \mathbb{R}^{K\times|\mathcal{U}|} \mapsto \|Vec(a)\|$, this is a compact Euclidean space. The rest of the proof follows the same lines as the proof of Proposition 1 and Glicksberg (1952).

Definition 3 *When the type space is finite, we define the pivotality at vote v for issue k corresponding to the symmetric mixed strategy equilibrium σ for the N-player game Γ^N*

$$\tilde{p}_k^N(v) = \mathbb{E}_\sigma\left[g\left(c\frac{1}{N}\left(\sum_{i=1}^{n} V_{ik} + v\right)\right)\right]$$

and the approximate pivotality is given by

$$p_k^N = \mathbb{E}_\sigma\left[g\left(c\frac{1}{N}\sum_{i=1}^{n} V_{ik}\right)\right].$$

Proposition 7 *In any sequence of mixed strategy equilibria*

$$\lim_{N\to\infty}\sup_{v\in[-1,1]}|\tilde{p}_k^N(v) - p_k^N| = 0$$

for all $k \in [K]$.

Proof does not substantially differ from the proof of Proposition 2.

Lemma 15 *Let σ^N be a symmetric mixed strategy equilibrium. For any $u \in \mathcal{U}$ there is a sequence $\epsilon_N \downarrow 0$ such that $\mathbb{P}\left[\|V_i - t(u, p^N)\| < \epsilon_N\right] = 1$ for all $N \in [N]$.*

Proof. In any equilibrium, the probability that a player plays an optimal action must be equal to 1, i.e. for all $i \in [N]$ and $k \in [K]$, when $U_i = u \in \mathcal{U}$,

$$V_{ik} = \frac{u_{ik}\mathbb{E}_{\sigma^N}\left[g\left(c\frac{1}{N}\sum_{i=1}^{N} V_{ik}\right)\right]}{\sqrt{\sum_{l=1}^{K} u_{il}^2\mathbb{E}_{\sigma^N}\left[g\left(c\frac{1}{N}\sum_{i=1}^{N} V_{il}\right)\right]^2}}$$

with probability equal to 1. Using the definitions of pivotality for mixed strategy equilibria (Definition 3), this can be written as:

$$V_{ik} = t(u, \tilde{p}_k^N(V_{ik})) \tag{18}$$

We know from Proposition 7 that $\tilde{p}_k^N$ converges to p_k^N uniformly so that by the same reasoning as in Lemma 9, on the probability 1 event that equation 18 holds, $V_{ik} - t(u, \tilde{p}_k^N(V_{ik})) < \epsilon_N$ for some sequence $\epsilon_N \downarrow 0$. ■

Lemma 16 *For constants $r \in [0, 0.5), d \in \mathbb{R}_{++}$ and $g(0) \in \mathbb{R}_{++}$ such that $-d^2 + d\sqrt{d^2+1} < 0.5$[13], consider the equation in x:*

$$-r + (1-r)\frac{x\,d}{\sqrt{g(0)^2 + x^2 d^2}} = 0.$$

There is a unique positive value $x^\star$ solving this equation. Iff $r > -d^2 + d\sqrt{d^2+1}$, then $x^\star > g(0)$. (This does not depend on the value of $g(0)$.)

Proof. Rearranging the equation: $x^2 d^2((1-r)^2 - r) = r^2 g(0)$, so $x^\star = \frac{1}{d}\frac{r}{\sqrt{1-2r}}g(0)$. So

$$\begin{aligned} x^\star > g(0) &\iff \\ \frac{r}{\sqrt{1-2r}} > d &\iff \\ r^2 + 2d^2 r - d^2 > 0.& \end{aligned}$$

The quadratic equation in r, $r^2+2d^2r-d^2 = 0$, has the solutions $r_1 = -d^2 - d\sqrt{d^2+1} < 0$ and $r_2 = -d^2 + d\sqrt{d^2+1} > 0$. Hence, since r is positive, $r^2+2d^2r-d^2 > 0$ for $r > -d^2+d\sqrt{d^2+1}$, so that by the sequence of equivalances above:

$$x^\star > g(0) \iff r > -d^2 + d\sqrt{d^2+1}$$

■

Proposition 9 *Suppose the type space and distribution is as described by Table 1. For any $c > 0$, for any $N \in \mathbb{N}$, for any equilibrium σ of Γ^N, let $\rho_\sigma^N := \mathbb{E}_\sigma\left[G\left(c\frac{1}{N}\sum_{i=1}^N V_{i2}\right)\right]$ denote the probability that $D_2 = 1$ is chosen in poll 2. If $-d^2 + d\sqrt{d^2+1} < 0.5$ and $r \in (-d^2 + d\sqrt{d^2+1}, 0.5)$, then for any $c > 0$ there is a convergent sequence $\{X^N\}_{N\in\mathbb{N}}$ of upper bounds such that in any sequence of equilibria $\{\sigma^N\}$:*

$$\rho_{\sigma^N}^N \leq X^N$$

and

$$\lim_{c\to\infty}\lim_{N\to\infty} X^N = 0.$$

Proof. The implementation probability ρ_σ^N depends on the average vote. We decompose the average vote into the votes cast by voters of type 1 and votes cast by type 2 or 3 voters, respectively: (For notational simplicity, I denote by $U_i = j$ the event that voter i is of type $j \in \mathbb{N}$.)

$$\frac{1}{N}\sum_{i=1}^N V_{i2} = \frac{1}{N}\sum_{i=1}^N V_{i2}\mathbb{1}\{U_i = 1\} + \frac{1}{N}\sum_{i=1}^N V_{i2}\mathbb{1}\{U_i \in \{2,3\}\}$$

[13]Values for d satisfying this condition exist, e.g. $d = 1$.

Regardless of N and c, voters of type 1 always vote $V_{i2} = -1$ in any equilibrium as this is always a dominant strategy given that G is strictly increasing. Thus,

$$\frac{1}{N}\sum_{i=1}^{N} V_{i2}\mathbb{1}\{U_i = 1\} \xrightarrow{p} -r \tag{19}$$

(where $r := \mathbb{P}[U_i = 1]$ as stated in the main text). When it comes to type 2 or 3 voters, we know from Lemma 15 that there is a sequence $\epsilon_N \downarrow 0$ so that

$$\mathbb{P}\left[|V_{i2} - t_2(U_i, p^N)| < \epsilon_N\right] = 1. \tag{20}$$

For any given c, let $\mathcal{Q}$ denote the set of fixed points of the associated ψ (as before). Define $\mathcal{Q}_1 := \{x \in \mathcal{P} | \exists y \in \mathcal{P} : (x, y) \in Q\}$ the projection of $\mathcal{Q}$ onto the first coordinate axis and $\mathcal{Q}_2$ its projection onto the second coordinate axis. Note that $\mathcal{Q}_1 = \{g(0)\}$ because $\psi_1(q) = \mathbb{E}\left[r\, t(u = 1, q) + \frac{1-r}{2}t(u = 2, q) + \frac{1-r}{2}t(u = 3, q)\right]$ and $t(1, q) = 0$ and $t(2, q) = -t(3, q)$ for all $q \in \mathcal{P}$. Thus, by Proposition 3, $\lim_{N\to\infty} p_1^N = g(0)$ in any sequence of equilibria. Proposition 3 also implies that $p_2^N < \sup \mathcal{Q}_2 + \eta_N$ for some sequence $\eta_N \downarrow 0$. For both $u \in \{2, 3\}$, $t_2(u, \cdot)$ is continuous and increasing in q_2. Thus, there is a sequence $\omega_N \downarrow 0$ such that

$$t_2(u, p^N) < t_2(u, (g(0), \sup Q_2)) + \omega_N$$

for all $u \in \{1, 2, 3\}$. We can combine this with equation 20 to deduce

$$\begin{aligned}
&\mathbb{P}\left[V_{i2} - t_2(U_i, (g(0), \sup \mathcal{Q}_2)) < \epsilon_N + \omega_N\right] \\
=&\mathbb{P}\left[V_{i2} - t_2(U_i, p^N) + t_2(U_i, p^N) - t(U_i, (g(0), \sup \mathcal{Q}_2)) < \epsilon_N + \omega_N\right] \\
\geq&\mathbb{P}\left[V_{i2} - t_2(U_i, p^N) < \epsilon_N\right] \\
=&1
\end{aligned}$$

(The inequality between the second and the third probability is due to the fact that the third event implies the second.) Therefore, with probability 1,

$$\begin{aligned}
&\frac{1}{N}\sum_{i=1}^{N} V_{i2}\mathbb{1}\{U_i \in \{2, 3\}\} \\
\leq&\frac{1}{N}\sum_{i=1}^{N}(t_2(U_i, (g(0), \sup \mathcal{Q}_2)) + \epsilon_N + \omega_N)\mathbb{1}\{U_i \in \{2, 3\}\} \\
\xrightarrow{p}&\mathbb{P}[U_i \in \{2, 3\}]\, t_2(2, (g(0), \sup \mathcal{Q}_2))
\end{aligned}$$

(Note that $t_2(2, \cdot) = t_2(3, \cdot)$.) Combining this result with $\frac{1}{N}\sum_{i=1}^{N} V_{i2}\mathbb{1}\{U_i = 1\} \xrightarrow{p} -r$, we conclude that for any $\epsilon > 0$,

$$\begin{aligned}
&\lim_{N\to\infty} \mathbb{P}\left[\frac{1}{N}\sum_{i=1}^{N} V_{i2} < -r + (1 - r)t_2(2, (g(0), \sup \mathcal{Q}_2)) + \epsilon\right] \\
=&\lim_{N\to\infty} \mathbb{P}\left[\frac{1}{N}\sum_{i=1}^{N} V_{i2} < \mathbb{E}\left[t_2(U_i, (g(0), \sup \mathcal{Q}_2))\right] + \epsilon\right] \\
=&1.
\end{aligned} \tag{21}$$

This holds for any sequence of equilibria. We can now proceed as in the proof of proposition 4, to show that $\limsup_{N\to\infty}\rho^N_{\sigma^N} \leq G\left(c\mathbb{E}\left[t_2(U_i,(g(0),\sup\mathcal{Q}_2))\right]\right)$: For any $\epsilon>0$, define $I_\epsilon := \mathbb{1}\{\frac{1}{N}\sum_{i=1}^N V_{i2} < \mathbb{E}\left[t_2(U_i,(g(0),\sup\mathcal{Q}_2))\right]+\epsilon\}$. Then

$$\lim_{N\to\infty}\mathbb{E}_{\sigma^N}\left[G\left(c\frac{1}{N}\sum_{i=1}^N V_{i2}\right)\right]-\mathbb{E}_{\sigma^N}\left[G\left(c\frac{1}{N}\sum_{i=1}^N V_{i2}\right)I_\epsilon\right]=\lim_{N\to\infty}\mathbb{E}_{\sigma^N}\left[G\left(c\frac{1}{N}\sum_{i=1}^N V_{i2}\right)(1-I_\epsilon)\right]$$
$$=0.$$

Therefore,

$$\limsup_{N\to\infty}\rho^N_{\sigma^N}=\limsup_{N\to\infty}\mathbb{E}_{\sigma^N}\left[G\left(c\frac{1}{N}\sum_{i=1}^N V_{i2}\right)I_\epsilon\right] \tag{22}$$

Further, because G is increasing, $G\left(c\frac{1}{N}\sum_{i=1}^N V_{i2}\right)\leq G\left(c\mathbb{E}\left[t_2(U_i,(g(0),\sup\mathcal{Q}_2))\right]+\epsilon\right)$ whenever $\frac{1}{N}\sum_{i=1}^N V_{i2}\leq\mathbb{E}\left[t_2(U_i,(g(0),\sup\mathcal{Q}_2))\right]+\epsilon$, i.e. whenver $I_\epsilon=1$. Thus

$$\mathbb{E}_{\sigma^N}\left[G\left(c\frac{1}{N}\sum_{i=1}^N V_{i2}\right)I_\epsilon\right]\leq G\left(c\mathbb{E}\left[t_2(U_i,(g(0),\sup\mathcal{Q}_2))\right]+\epsilon\right)\mathbb{E}_{\sigma^N}\left[I_\epsilon\right]$$

so that

$$\limsup_{N\to\infty}\mathbb{E}_{\sigma^N}\left[G\left(c\frac{1}{N}\sum_{i=1}^N V_{i2}\right)I_\epsilon\right]\leq\limsup_{N\to\infty}G\left(c\mathbb{E}\left[t_2(U_i,(g(0),\sup\mathcal{Q}_2))\right]+\epsilon\right)\mathbb{E}_{\sigma^N}\left[I_\epsilon\right].$$

From equation 21 we know that $\lim_{N\to\infty}\mathbb{E}_{\sigma^N}\left[I_\epsilon\right]=1$ for any $\epsilon>0$. Thus, given equation 22 in any sequence of equilibria $\{\sigma^N\}_{N\in\mathbb{N}}$

$$\limsup_{N\to\infty}\rho^N_{\sigma^N}\leq G\left(c\mathbb{E}\left[t_2(U_i,(g(0),\sup\mathcal{Q}_2))\right]\right).$$

Hence for any $c>0$ there is a sequence X^N with

$$\lim_{N\to\infty}X^N=G\left(c\mathbb{E}\left[t_2(U_i,(g(0),\sup\mathcal{Q}_2))\right]\right) \tag{23}$$

such that $\rho^N_{\sigma^N}\leq X^N$ in any sequence of equilibria $\{\sigma^N\}^N$ and for any $N\in\mathbb{N}$. $x\mapsto t_2(u,(g(0),x)$ is continuous for any $u\in\{1,2,3\}$ and bounded. Therefore, Lebesgue's Dominated Convergence Theorem can be applied to show that $x\mapsto\mathbb{E}\left[t_2(U_i,(g(0),x))\right]$ is continuous. Since G is continuous, $x\mapsto G\left(c\mathbb{E}\left[t_2(U_i,(g(0),x))\right]\right)$ is continuous. Now define the function $\nu:\mathbb{R}_{++}\mapsto\mathbb{R}_{++}$ by

$$\nu(x)=\sup\{s\in\mathbb{R}_{++}:\exists q\in\mathcal{P}^2:q_2=s\text{ and }q_k=g(x\mathbb{E}\left[t_k(U_i,q)\right])\,\forall k\in\{1,2\}\},$$

i.e. $\nu(x)$ is $\sup\mathcal{Q}_2$ when $c=x$ so that $G\left(c\mathbb{E}\left[t_2(U_i,(g(0),\sup\mathcal{Q}_2))\right]\right)=G\left(c\mathbb{E}\left[t_2(U_i,(g(0),\nu(c)))\right]\right)$. Suppose $\lim_{x\to\infty}\nu(x)=0\in\mathbb{R}_+$. Then $\lim_{x\to\infty}\mathbb{E}\left[t_2(U_i,(g(0),\nu(x)))\right]=-r$ and $\lim_{x\to\infty}x\mathbb{E}\left[t_2(U_i,(g(0),\nu(x$ $-\infty$ and so $\lim_{x\to\infty}G(x\mathbb{E}\left[t_2(U_i,(g(0),\nu(x)))\right])=0$. Next, we show that $\lim_{x\to\infty}v(x)=0$ if

$r > -d^2 + d\sqrt{d^2+1}$: We know that $\mathcal{Q}_1 = \{g(0)\}$ because $\psi_1(q) = g(0)$ for all $q \in \mathcal{P}$ (and ψ must have at least one fixed point by Lemma **??**). The set of second coordinates of the fixed points of ψ is the set of solutions of the condition

$$\psi_2((g(0), q_2)) \equiv g\left(c\mathbb{E}\left[t_2(U, (g(0), q_2))\right]\right) \equiv g\left(c\left(-r + (1-r)\frac{q_2 d}{\sqrt{g(0)^2 + q_2^2 d^2}}\right)\right) = q_2$$

Let $x^\star$ be the (unique positive) solution for x of the equation

$$-r + (1-r)\frac{x}{\sqrt{g(0)^2 + x^2}} = 0.$$

From the previous lemma (Lemma **??**), we know that $x^\star > g(0) \equiv \psi_2(g(0), x^\star)$ if and only if $r > -d^2 + d\sqrt{d^2+1}$. We show that in this case, $\lim_{c\to\infty} \sup \mathcal{Q}_2 = 0$: As g is decreasing in the absolute value of its input (by assumption), $x^\star$ is a global maximum of ψ_2. As a consequence, there cannot be a fixed point q of ψ with $q_2 > \psi_2(g(0), x^\star)$. Furthermore, by assumption, $\lim_{z\to-\infty} g(z) = 0$, which means that for any $\epsilon > 0$ there is $z_\epsilon (< 0)$ such that for any $y < z_\epsilon$, $g(y) < \epsilon$. Thus, if c is so large that $c\mathbb{E}\left[t_2(U, (g(0), \psi(g(0), x^\star))\right] < z_\epsilon$, then $\psi_2(g(0), x) \equiv g(c\mathbb{E}\left[t_2(U, (g(0), x))\right] < \epsilon$ for any $x \leq \psi((g(0), x^\star))$ (keeping in mind that $\mathbb{E}\left[t_2(U, (g(0), x))\right]$ is increasing in x). In other words, for any $\epsilon > 0$, there is $c_\epsilon \left(:= z_\epsilon / \mathbb{E}\left[t_2(U, (g(0), \psi_2(x^\star))\right]\right)$, such that for any $c > c_\epsilon$ and any $x \leq \psi((g(0), x^\star))$, $\psi_2((g(0), x^\star)) < \epsilon$ so that there cannot be a fixed point q of ψ with $q_2 \geq \epsilon$. We know however that a fixed point must exist (i.e. $\mathcal{Q}_2$ is non-empty) so that we can write $\lim_{c\to\infty} \sup \mathcal{Q}_2 = 0$. Returning to equation 23 we thus conclude

$$\lim_{c\to\infty} \lim_{N\to\infty} X^N = 0$$

■

B Summary of Notation

Model setup

- K issues: $D_1, ...D_K \in \{-1, 1\}$
- N voters. Let $n = N - 1$.
- Voter i has utility vector $U_i \sim F$. $\mathcal{U} := supp(F) \subseteq \mathbb{R}^K$. Utility vectors are drawn i.i.d. (across voters). $\mathcal{U} = \bigtimes_{k=1}^K \mathcal{U}_k$. $\mathcal{U}_{-l} = \bigtimes_{k=1, k\neq l}^K \mathcal{U}_k$.
- Each voter has 1 vote credit which they use to buy a vote vector V_i realizing in $\mathbb{R}^K$. Budget constraint: $\|V_i\|^2 \leq 1$. That is, each player has action set $\mathcal{A} := \{x \in \mathbb{R}^K : \|x\| \leq 1\}$. Let $V := (V_1', ..., V_N')'$.
- Collective decision function: $\mathbb{P}\left[D_k = 1 | \frac{1}{N}\sum_{k=1}^K V_{ik} = cx\right] = G(x)$ is smooth function of the average vote. Let $g := G'$.
- Identical payoff functions: $\Pi : \mathcal{U} \times \mathcal{A}^N \mapsto \mathbb{R}$. $\Pi(u, v_1, ..., v_N) = \sum_{k=1}^K u_k G\left(c\frac{1}{N}\sum_{i=1}^N v_{ik}\right)$.

Definitions and notation

- Distributional strategy: Measure μ on $\mathcal{U} \times A$ such that for any subset Q of $\mathcal{U}$, $\mu(Q \times \mathcal{A}) = F(Q)$. Payoff when players play distributional strategies $\mu_1, .., \mu_N$: $\hat{\Pi}(\mu_1, ..., \mu_N) = \mathbb{E}_{\mu_1 \times ... \times \mu_N} [\Pi(U, V_1, ..., V_N)]$.
- Mixed strategy: function $\mathcal{U} \mapsto \Delta\mathcal{A}$. Denote the set of mixed strategies by $\mathcal{M}$.
- Interim expected payoff function

$$\bar{\Pi} : \mathcal{U} \times A \times \mathcal{M}^{N-1} \mapsto \mathbb{R},$$

$$\bar{\Pi}(u, v, \sigma_{-i}) = \mathbb{E}_{\sigma_{-i}} [\Pi(u, V_1, ..., V_{i-1}, v, V_{i+1}, ...V_N)] .$$

- Best responses of a player with preference profile $u \in \mathcal{U}$ to a profile of mixed strategies for all other players, $\sigma_{-i} \in \mathcal{M}^{N-1}$

$$BR(u, \sigma_{-i}) = \arg\max_{v \in \mathcal{A}} \bar{\Pi}(u, v, \sigma_{-i})$$

- Projections of best responses onto kth coordinate axis: $BR_k(u, \sigma_{-i}) := \{a \in [-1, 1] : \exists b \in BR(u, \sigma_{-i}) : b_k = a\}$.
- Ex-ante Bayes Nash equilibrium: see text above
- Interim Bayes Nash equilibrium: standard BNE
- $s_k(u, q) = \frac{u_k q_k}{\sqrt{\sum_{l=1}^{K} u_l^2 {q_l}^2}}$ and $s(u, 0) = s(0, q) = 0$ for all $u \in \mathcal{U}$, $q \in \mathbb{R}^K$ and $k \in [K]$.
- $\mathcal{P} := [\min_{x \in [-1,1]} g(x), \max_{x \in [-1,1]} g(x)]^K$
- $\psi^N : \mathcal{P} \mapsto \mathcal{P}$ with $\psi_k^N(q) = \mathbb{E}\left[g\left(c\frac{1}{N}\sum_{i=1}^{n} s_k(U_i, q)\right)\right]$
- $\psi : \mathcal{P} \mapsto \mathcal{P}$ with $\psi(q) = g\left(c\mathbb{E}\left[s(U_i, q)\right]\right)$
- $\tilde{p}_k^N(u) = \mathbb{E}\left[g\left(c\frac{1}{N}\left(\sum_{i=1}^{n} s_k^N(U_i) + s_k^N(u)\right)\right)\right]$
- $p_k^N = \mathbb{E}\left[g\left(c\frac{1}{N}\sum_{i=1}^{n} s_k^N(U_i)\right)\right]$